SENIOR SECONDARY COURSE

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MATHEMATICS

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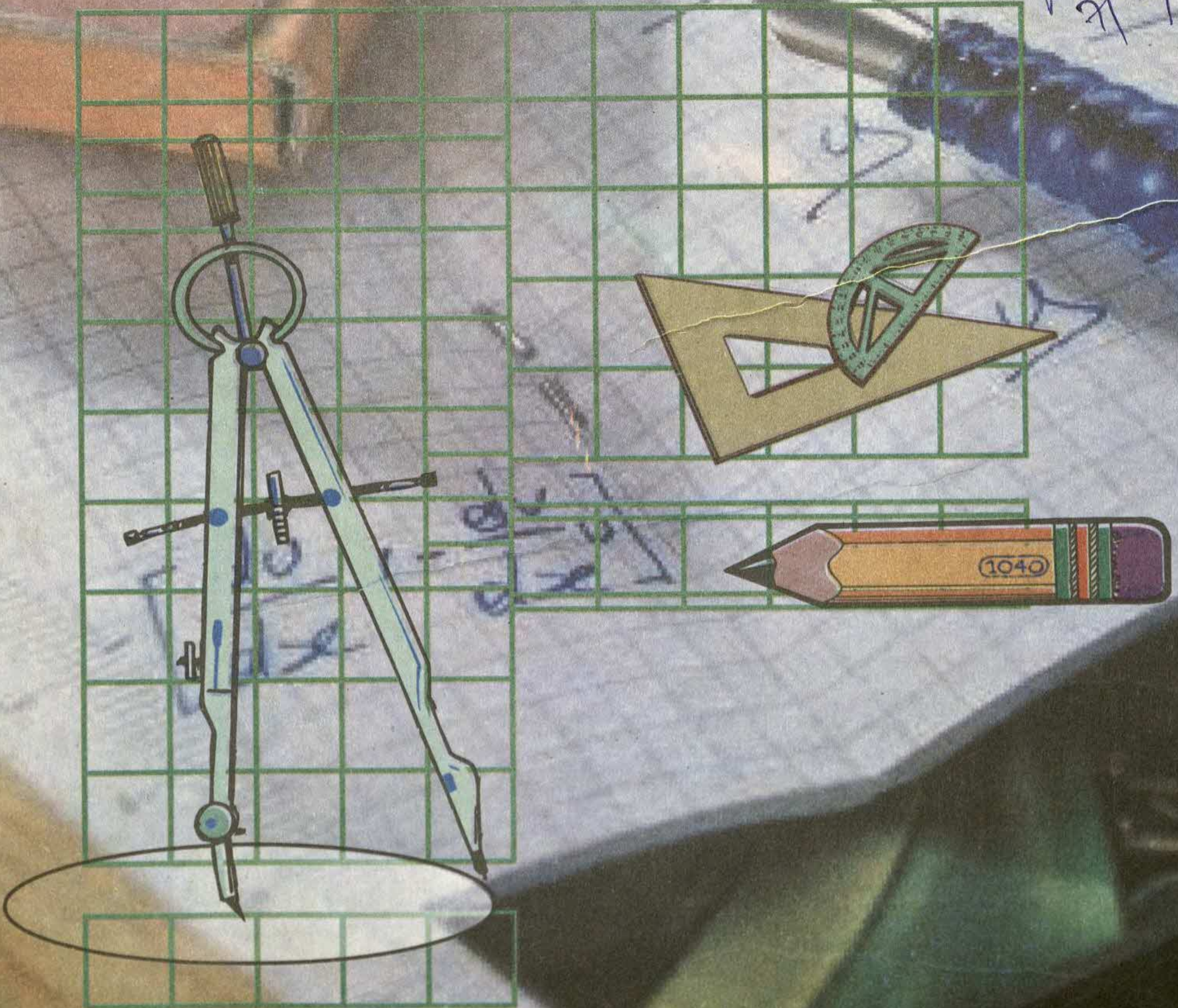

NATIONAL INSTITUTE OF OPEN SCHOOLING

SENIOR SECONDARY COURSE
MATHEMATICS

1

(CORE MODULES)

Designed and developed
by
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A-24/25,INSTITUTIONAL AREA
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FOREWORD

Dear Learner,

We are very happy that you have joined NIOS and decided to become an Open and Distant Learner.

NIOS has brought out its revised Senior Secondary Course material. This course has been designed in a **modular format** in the sense that the content is divided into different modules. These modules are made up of a number of lessons. The modules are self contained and you can pick up any of the modules first that interests you. However, we would like you to proceed as the modules have been arranged because there are references and cross references to other lessons.

One of the main features of this course is the division of the modules into **core** and **optional** modules. The core modules are compulsory for all learners, but you can choose any **one** of the two optional modules. For example, in Geography, you can pick up either Local Area Planning **or** Tourism Geography; in Business Studies, you can study either Wage Employment in Business **or** Self Employment in Business. Each subject offers two optional modules. The idea is to allow you to study what interests you more even in one particular subject. This is something unique to the NIOS courses of study. You will not find such **choice** elsewhere.

This course is revised on the lines of the guidelines contained in the National Curriculum Framework 2005. We have tried to make the material as **activity based** as possible. We believe that you learn more when you do something with your own hands rather than just passive reading. We have made efforts to keep the **language of the study material simple** to facilitate you to understand the content easily.

The examples that we have chosen and used in the course material are from daily life to enable you to relate easily what is new to what you already know. Through the Self Learning Materials, we are trying to help you to construct your own knowledge so that you learn by understanding and not just by memorising everything.

Another important feature of this learning material is integration of Adolescence Education issues with the learning content. Realising that development of life skills like self awareness, critical thinking, negotiation and communication skills is important, we have used different opportunities to build desirable skills in the lessons.

The study materials developed by NIOS are **self-learning materials**. You are supposed to read and work on your own. Unlike a textbook, you do not need a teacher to tell you what to do.

With our good wishes, start studying, do what you are told to do, attempt all activities, answer the intext questions, check your answers from the answers given, learn each topic well and be a **successful self learner**.

Chairman, NIOS

A NOTE FROM THE DIRECTOR

Dear Learner,

The Academic Department at the National Institute of Open Schooling tries to bring you new programmes in accordance with your needs and requirements. After making a comprehensive study, we found that our curriculum is more functional, related to life situations and simple. The task now was to make it more effective and useful for you. We invited leading educationists of the country and under their guidance, we have been able to revise and update the Senior Secondary curriculum in the subject of Mathematics in the light of National Curriculum Framework (NCF)-2005.

At the same time, we have also removed old, outdated information and added new, relevant things and tried to make the learning material attractive and appealing for you.

I hope you will find the new material interesting and exciting . Any suggestions for further improvement are welcome.

Let me wish you all a happy and successful future.

How to Study in the National Institute of Open Schooling

You should note that by taking admission in the National Institute of Open Schooling (NIOS), you have entered in a system where studying is different than a normal school.

You Are a Self Learner Now

In a school, a teacher is always available to take class, clarify doubts, guide and encourage. There you also must be discussing with your peer-group, going to the library, doing practical, participate in co-curricular activities, watching educational TV, Radio programmes etc. All this was also contributing to your learning.

However, in the NIOS, there is no teacher available and you have to learn on your own. It means that you have become a self-learner. Responsibilities of a self-learner are much more than a normal learner who is dependent on a teacher; but at the same time it is also challenging.

Here it is solely you, who is responsible for your learning. It means that you have to organize your study, learn regularly, keep up your motivation and achieve your goal.

Understanding Your Learning Material

The NIOS will help you by providing learning material, part of which is in your hands at this moment.

We call them learning materials because these are different from the textbooks you have read in your schools. You will find that the contents, concepts and topics have been explained here in a way the teacher does in a classroom. You will also find examples and illustrations to help you understand the things properly.

This is the reason you will find them bulky, but let this shall not frighten you.

You will find a few sections in your lessons. Let us know the purpose of these:

First paragraph of each lesson will introduce you to the topic.

Objectives: Here you will find the list of objectives, which you shall achieve after learning the lesson. You can actually check whether you have achieved these or not because these are presented in measurable terms.

Check Your Progress : These will appear after every section of the lesson. They will contain very short answer, short answer and long answer type questions based on each lesson. This will help you whether you have learnt the section or not. Your will find the key to these questions at the end of the lesson. If you are able to answer the questions, then you can proceed further otherwise you should learn the section again.

Let us Sum up: Here you will find the summary of the main points of the lesson for recapitulation and revision.

Supportive Web Sites: You may explore in these web sites for extended learning

Terminal Exercise: Here short answer and long answer type questions are given to help you foster your learning and give you an opportunity to practice for examination.

Answers: Answers of all "Check Your Progress" and "Terminal Exercise" are given at the end of each lesson. Some hints are also given in case of a difficult question.

Apart from the learning material, you will also get, sample question papers, previous year question papers, etc., for help in your studies and for practice for examination.

Personal Contact Programmes

A few sessions/ classes would be provided to you at the center. You should, however, note that those are not meant for teaching you as it happens in a normal school. Here you will have an opportunity to clarify your doubts, solve your problems and get guidance and advice on your study. So go to the classes well prepared to get maximum benefit.

Use of audio and video programmes

NIOS has also developed a few audio and video programmes, which will be very interesting for you and will help you in your studies. You can take copies of these on loan from your Center.

Planning and Organizing Your Study

Let me also give you some tips for planning and organizing your study.

First of all, you need to understand that there is no substitute for hard work. Harder you work better you achieve. Also, there are no short cuts to success. In order to succeed in life and get benefit of your learning , you need to learn.

As you must have understood by how, the NIOS provides a lot of freedom and flexibilities in your study. For example, there is no need to take examination in all the subjects simultaneously. So, first of all, think about the time available with you and decide whether you wish to study all the subjects together or you wish to learn them one after the other. Just gambling in all the subjects will bring you to a situation where you will not be able concentrate on any subject.

Now fix up a time for study, evening, morning or daytime which ever is convenient. Draw a timetable giving justified time to the subjects you have taken up for study and follow this as far as possible. While you study, underline the concepts you feel are important. You should study from NIOS learning material. In addition you may read other books, if you have time. However, for your purpose, this material will be sufficient. Keep a copy each for the subjects you are preparing. Note down the points you did not understand. Discuss these with your parents, friends or teachers at your study centre.

Solve all the exercises, and practice exercise of each module appearing in the material. This will not only help you in learning, but also provide practice for your examination. You may also like to solve sample papers and previous years' question papers. Show your answers to your parents and friends and discuss with them.

These were a few hints to help you. You may find some other techniques which may work better with you. Follow those, if you like. I am sure you will be successful in your endeavor.

A Word With You

Dear Learner,

Welcome to the Senior Secondary course. It gives me great pleasure to know that you have opted for Mathematics as one of your subjects of study. Have you ever thought as to why we study Mathematics? Can you think one day when you have not counted something or used Mathematics?
Probably not.

Mathematics is the base of human civilization. From cutting vegetables to arranging books on the shelves, from tailoring cloth to motion of planets –Mathematics applies everywhere. In fact, everything we do in our daily life is governed by Mathematics. Mathematics can be broadly defined as the scientific study of quantities, including their relationships, operations and their measurements expressed by numbers and symbols. The mathematicians claim that the learning of Mathematics can be real fun. It only requires complete concentration and love for Mathematics.

The present curriculum has **six core** modules namely, Algebra, Coordinate Geometry, Sequences and Series, Functions, Calculus and Statistics. Besides these, you have to choose any **one** of the optional modules namely, Vectors and Three Dimensional Geometry, and Mathematics for Commerce, Economics and Business. There will be **two books** to cover the core modules and **one** book for the optional modules.

Book 1 contains the first three core modules.

In the module on Algebra, you will be introduced to complex numbers and quadratic equations and various operations with them. This module will show you how to solve a system of linear equations with the help of matrices and determinants.

The second module on Coordinate Geometry will introduce you to various forms of straight lines, circles, and conic sections.

In our day to day life, we often come across different patterns of figures, pictures, numbers etc. If you observe these patterns carefully, you will find that they follow certain rules. You will be encouraged to recognize different patterns of numbers and then introduced to the concept of Sequences and Series in the third module.

We would suggest that you go through all the solved examples given in the learning material and then try to solve independently all questions included in "**Check Your Progress**" and "**Terminal Exercise**" given at the end of each lesson.

This book contains three **"Questions For Practice"** pages for self-check. Try to solve all of them and get it checked by the subject teacher at your study centre so that you get positive feedback about your performance. Your success is reward to us.

If you face any difficulty please do write to us. Your suggestions are also welcome.

Wishing you all success.

Yours

(Suvendu Sekhar Das)
Academic Officer (Mathematics)

An Overview of the Learning Material

CORE MODULES

ALGEBRA

- Complex Numbers
- Quadratic Equations
- Matrices
- Determinants and their Applications
- Inverse of a Matrix and its Application
- Linear Inequations and their Applications
- Permutations and Combinations
- Binomial Theorem

SEQUENCES AND SERIES

- Arithmetic and Geometric Progressions
- Special Types of Series

CALCULUS

- Limit and Continuity
- Differentiation
- Differentiation of Trigonometric Functions
- Differentiation of Exponential and Logarithmic Functions
- Tangents and Normals
- Maxima and Minima
- Integration
- Definite Integrals
- Differential Equations

COORDINATE GEOMETRY

- Cartesian System of Coordinates
- Straight Lines
- Circles
- Conic Sections

FUNCTIONS

- Sets, Relations and Functions
- Trigonometric Functions-I
- Trigonometric Functions-II
- Inverse Trigonometric Functions
- Relations between Sides and Angles of a Triangle

STATISTICS

- Measures of Dispersion
- Random Experiments and Events
- Probability

OPTIONAL MODULES

(You have to choose any one of the following modules)

VECTORS AND THREE DIMENSIONAL GEOMETRY

- Vectors
- Introduction to Three Dimensional Geometry
- The Plane
- The Straight Line
- The Sphere

MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS

- Shares and Debentures
- Index Numbers
- Insurance
- Indirect Taxes
- Application of Calculus in Commerce and Economics

CONTENTS

	Lesson No.	Title	Page No.
MODULE- I		**ALGEBRA**	**1 - 306**
	1	Complex Numbers	1
	2	Quadratic Equations	39
	3	Matrices	59
	4	Determinants and their Applications	101
	5	Inverse of a Matrix and its Application	147
	6	Linear Inequations and their Applications	189
	7	Permutations and Combinations	237
	8	Binomial Theorem	265
		Questions For Practice-Algebra	305
MODULE -II		**COORDINATE GEOMETRY**	**307-404**
	9	Cartesian System of Coordinates	307
	10	Straight Lines	339
	11	Circles	379
	12	Conic Sections	391
		Questions For Practice-Coordinate Geometry	403
MODULE -III		**SEQUENCES AND SERIES**	**405-454**
	13	Arithmetic and Geometric Progressions	405
	14	Special Types of Series	441
		Questions For Practice -Sequences and Series	453
		Curriculum ***Feedback Form***	**455-466**

T.M.A. I, II, III

MODULE-I

ALGEBRA

1. Complex Numbers
2. Quadratic Equations
3. Matrices
4. Determinants and Their Applications
5. Inverse of a Matrix and its Application
6. Linear Inequations and Their Applications
7. Permutations and Combinations
8. Binomial Theorem

1

COMPLEX NUMBERS

We started our study of number systems with the set of natural numbers, then the number zero was included to form the system of whole numbers;and in the next phase negative numbers were defined. Thus, we extended our number system to whole numbers and integers.

To solve the problems of the type $p \div q$, we included rational numbers in the system of integers. The system of rational numbers have been extended further to irrational numbers as all lengths cannot be measured in terms of lengths expressed in rational numbers. Rational and irrational numbers taken together are termed as real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. There are no real numbers which satisfy the equation $x^2 + 1 = 0$ or $x^2 = -1$. In order to solve such equations, i.e., to find square roots of negative numbers, we extend the system of real numbers to a new system of numbers known as complex numbers. In this lesson, the learner will be acquainted with complex numbers, their representation and algebraic operations on complex numbers.

OBJECTIVES

After studying this lesson, you will be able to:

- describe the need for extending the set of real numbers to the set of complex numbers;
- define a complex number and cite examples;
- identify the real and imaginary parts of a complex number;
- state the condition for equality of two complex numbers;
- recognise that there is a unique complex number $x + iy$ associated with the point $P(x, y)$ in the Argand Plane and vice-versa;
- define and find the conjugate of a complex number;
- define and find the modulus and argument of a complex number;

- represent a complex number in the polar form;
- perform algebraic operations (addition, subtraction, multiplication and division) on complex numbers;
- state and use the properties of algebraic operations (closure, commutativity, associativity, identity, inverse and distributivity) of complex numbers; and
- state and use the following properties of complex numbers in solving problems:

(i) $|z| = 0 \Leftrightarrow z = 0$ and $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

(ii) $|z| = |-z| = |\bar{z}|$ (iii) $|z_1 + z_2| \le |z_1| + |z_2|$

(iv) $|z_1 \cdot z_2| = |z_1| . |z_2|$ (v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

EXPECTED BACKGROUND KNOWLEDGE

- Properties of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of points in a plane.

1.1 COMPLEX NUMBERS

Consider the equation $x^2 + 1 = 0$. ...(A)

This can be written as $x^2 = -1$

or $x = \pm\sqrt{-1}$

But there is no real number which satisfies $x^2 = -1$.In other words, we can say that there is no real number whose square is -1.In order to solve such equations, let us imagine that there exists a number 'i' which equal to $\sqrt{-1}$.

In 1748, a great mathematician, L. Euler named the number 'i' as *Iota* whose square is -1. This *Iota or 'i'* is defined as imaginary unit. With the introduction of the new symbol 'i', we can interpret the square root of a negative number as a product of a real number with i

Therefore, we can denote the solution of (A) as $x = \pm i$

Thus, $-4 = 4(-1)$

$$\therefore \quad \sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2 . 2^2} = 2i$$

Conventionally written as 2i.

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are examples of complex numbers.

Consider another quadratic equation:

$$x^2 - 6x + 13 = 0$$

This can be solved as under:

$$(x-3)^2 + 4 = 0$$

or, $(x-3)^2 = -4$

or, $x - 3 = \pm 2i$

or, $x = 3 \pm 2i$

We get numbers of the form $x + yi$ where x and y are real numbers and $i = \sqrt{-1}$.

Any number which can be expressed in the form $a + bi$, where a,b are real numbers and $i = \sqrt{-1}$, is called a complex number.

A complex number is, generally, denoted by the letter z.

i.e. $z = a + bi$, 'a' is called the real part of z and is written as Re (a+bi) and 'b' is called the imaginary part of z and is written as Imag (a + bi).

If $a = 0$ and $b \neq 0$, then the complex number becomes b*i* which is a purely imaginary complex number.

$-7i$, $\frac{1}{2}i$, $\sqrt{3}i$ and πi are all examples of purely imaginary numbers.

If $a \neq 0$ and $b = 0$ then the complex number becomes 'a' which is a real number.

5, 2.5 and $\sqrt{7}$ are all examples of real numbers.

If $a = 0$ and $b = 0$, then the complex number becomes 0 (zero). Hence the real numbers are particular cases of complex numbers.

Example 1.1 Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}.\sqrt{-4}$

Solution: (i) $\sqrt{-36} = \sqrt{36(-1)} = 6i$

(ii) $\sqrt{25}.\sqrt{-4} = 5 \times 2i = 10i$

1.2 POSITIVE INTEGRAL POWERS OF i

We know that

$$i^2 = -1$$

$$i^3 = i^2.i = -1.i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$i^5 = (i^2)^2.i = 1.i = i$$

$$i^6 = (i^2)^3 = (-1)^3 = -1$$

$$i^7 = (i^2)^3(i) = -i$$

$$i^8 = (i^2)^4 = 1$$

Thus, we find that any higher powers of 'i' can be expressed in terms of one of four values i, –1, –i, 1

If n is a positive integer such that $n>4$, then to find i^n, we first divide n by 4.

Let m be the quotient and r be the remainder.

Then $n = 4m + r$. where $0 \leq r < 4$.

Thus,

$$i^n = i^{(4m+r)} = i^{4m}.i^r$$

$$= (i^4)^m.i^r$$

$$= i^r \ (\because i^4=1)$$

Note : For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when atleast one of a and b is either 0 or positive.

In fact $\sqrt{-a} \times \sqrt{-b}$

$$= i\sqrt{a} \times i\sqrt{b} = i^2\sqrt{ab}$$

$= -\sqrt{ab}$ where a and b are positive real numbers.

Example 1.2 Find the value of $1 + i^{10} + i^{20} + i^{30}$

Solution: $1 + i^{10} + i^{20} + i^{30}$

$$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15}$$

$$= 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$$

$$= 1 + (-1) + 1 + (-1)$$

$$= 1 - 1 + 1 - 1$$

$$= 0$$

Thus, $1 + i^{10} + i^{20} + i^{30} = 0$.

Example 1.3 Express $8i^3 + 6i^{16} - 12i^{11}$ in the form of a + bi

Solution: $8i^3 + 6i^{16} - 12i^{11}$ can be written as $8(i^2).i + 6(i^2)^8 - 12(i^2)^5.i$

$$= 8(-1).i + 6(-1)^8 - 12(-1)^5.i$$

$$= -8i + 6 - 12(-1).i$$

$$= -8i + 6 + 12i$$

$$= 6 + 4i$$

which is of the form of a + bi where 'a' is 6 and 'b' is 4.

CHECK YOUR PROGRESS 1.1

1. Simplify each of the following using 'i'.

   (a) $\sqrt{-27}$ (b) $-\sqrt{-9}$ (c) $\sqrt{-13}$

2. Express each of the following in the form of a + bi

   (a) 5 (b) $-3i$ (c) 0

3. Simplify $10i^3 + 6i^{13} - 12i^{10}$

4. Show that $i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0$ for all $m \in N$

1.3 CONJUGATE OF A COMPLEX NUMBER

Consider the equation:

$$x^2 - 6x + 25 = 0 \quad \ldots\ldots \quad (i)$$

or, $(x - 3)^2 + 16 = 0$

or, $(x - 3)^2 = -16$

or, $(x - 3) = \pm\sqrt{-16} = \pm\sqrt{16.(-1)}$

or, $x = 3 \pm 4i$

The roots of the above equation (i) are $3 + 4i$ and $3 - 4i$.

Consider another equation:

$$x^2 + 2x + 2 = 0 \quad \ldots \quad (ii)$$

or, $(x + 1)^2 + 1 = 0$

or, $(x + 1)^2 = -1$

or, $(x + 1) = \pm\sqrt{-1} = \pm i$

or, $x = -1 \pm i$

The roots of the equation (ii) are $-1+i$ and $-1-i$.

Do you find any similarity in the roots of (i) and (ii)?

The equations (i) and (ii) have roots of the type $a + bi$ and $a - bi$. Such roots are known as conjugate roots and read as $a + bi$ is conjugate to $a - bi$ and vice-versa.

The complex conjugate (or simply conjugate) of a complex number $z = a + bi$ is defined as the complex number $a - bi$ and is denoted by $\bar{z}$.

Thus, if $z = a + bi$ then $\bar{z} = a - bi$.

Note : The conjugate of a complex number is obtained by changing the sign of the imaginary part.

Following are some examples of complex conjugates:

(i) If $z = 2 + 3i$, then $\bar{z} = 2 - 3i$

(ii) If $z = 1-i$, then $\bar{z} = 1 + i$

(iii) If $z = -2 + 10i$, then $\bar{z} = -2-10i$

1.3.1 PROPERTIES OF COMPLEX CONJUGATES

(i) If z is a real number, then $z = \bar{z}$ i.e., the conjugate of a real number is the number itself.

For example, let $z = 5$

This can be written as

$$z = 5 + 0i$$

$$\therefore \quad \bar{z} = 5 - 0i = 5$$

$$\therefore \quad z = 5 = \bar{z}.$$

(ii) If z is a purely imaginary number then $\bar{z} = -z$

For example, if $z = 3i$

This can be written as

$$z = 0 + 3i$$

$$\therefore \quad \bar{z} = 0 - 3i = -3i$$

$$= -z$$

$$\therefore \quad \bar{z} = -z.$$

(iii) Conjugate of the conjugate of a complex number is the number itself.

i.e., $\overline{(\overline{z})}=z$

For example, if $z=a+bi$ then

$$\overline{z}=a-bi$$

Again $\overline{(\overline{z})}=\overline{(a-bi)}=a+bi$

$$=z$$

$$\therefore \overline{(\overline{z})}=z$$

Example 1.4 Find the conjugate of each of the following complex numbers:

(i) $3-4i$ (ii) $2i$

(iii) $(2+i)^2$ (iv) $\dfrac{i+1}{2}$

Solution : (i) Let $z=3-4i$

then $\overline{z}=\overline{(3-4i)}=3+4i$

Hence, $3+4i$ is the conjugate of $3-4i$.

(ii) Let $z=2i$ or $0+2i$

then $\overline{z}=\overline{(0+2i)}=0-2i$

Hence, $-2i$ is the conjugate of $2i$.

(iii) Let $z=(2+i)^2$

i.e. $z=(2)^2+(i)^2+2(2)(i)$

$=4-1+4i$

$=3+4i$

Then $\overline{z}=\overline{(3+4i)}=3-4i$

Hence, $3-4i$ is the conjugate of $(2+i)^2$

(iv) Let $z=\dfrac{i+1}{2}=\dfrac{1}{2}+\dfrac{1}{2}i$

then $\bar{z} = \overline{\left(\frac{1}{2}+\frac{1}{2}i\right)} = \frac{1}{2}-\frac{1}{2}i$

Hence, $\frac{1}{2}-\frac{i}{2}$ or $\frac{-i+1}{2}$ is the conjugate of $\frac{i+1}{2}$

1.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

Let z = a + bi be a complex number. Let two mutually perpendicular lines xox' and yoy' be taken as x-axis and y-axis respectively, O being the origin.

Let P be any point whose coordinates are (a,b). We say that the complex number z = a + bi is represented by the point P (a , b) as shown in Fig. 1.1

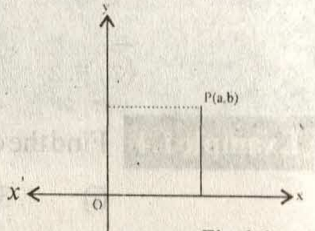

Fig. 1.1

If b = 0, then z is real and the point representing complex number z = a + 0i is denoted by (a, 0). This point (a, 0) lies on the x-axis.

So, xox' is called the real axis. In the Fig. 1.2 the point Q (a, 0) represents the complex number z = a + 0i.

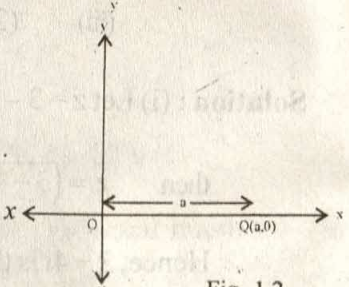

Fig. 1.2

If a = 0, then z is purely imaginary and the point representing complex number z = 0 + bi is denoted by (0, b). The point (0, b) lies on the y-axis.

So, yoy' is called the imaginary axis. In Fig. 1.3, the point R (0, b) represents the complex number z = 0 + bi.

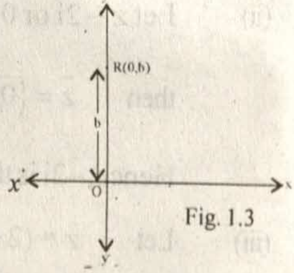

Fig. 1.3

The plane of two axes representing complex numbers as points is called the complex plane or Argand Plane.

The diagram which represents complex number in the Argand Plane is called Argand Diagram.

Example 1.5

Represent complex numbers 2 + 3i and 3 + 2i in the same Argand Diagram.

Solution :

1. 2+3i is represented by the point A (2,3)
2. 3+2i is represented by the point B (3, 2) Clearly, the points A and B are different

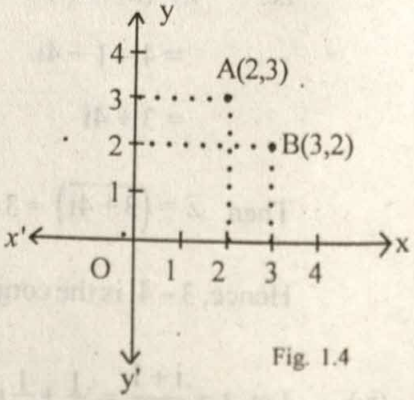

Fig. 1.4

Example1.6

Represent complex numbers 2 + 3i and –2 –3i in the same Argand Diagram.

Solution:

1. 2 + 3i is represented by the point P (2, 3)
2. –2 –3i is represented by the point Q (–2,–3)

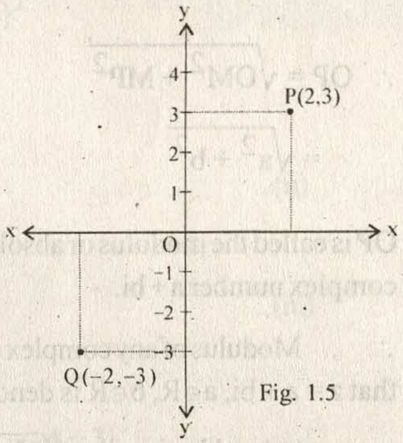

Fig. 1.5

Points P and Q are different and lie in the I quadrant and III quadrant respectively.

Example1.7

Represent complex numbers 2 + 3i and 2 – 3i in the same Argand Diagram

Solution:

1. 2 + 3i is represented by the point R(2, 3)
2. 2 – 3i is represtned by the point S(2, –3)

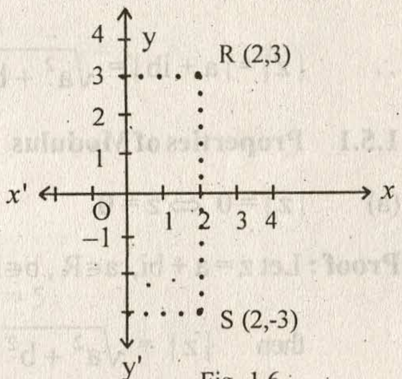

Fig. 1.6

Example1.8

Represent complex numbers 2 + 3i, –2–3i, and 2–3i in the same Argand Diagram

Solution:

(a) 2+3i is represented by the point P (2, 3)

(b) –2–3i is represented by the point Q (–2,–3)

(c) 2–3i is represented by the point R (2, –3)

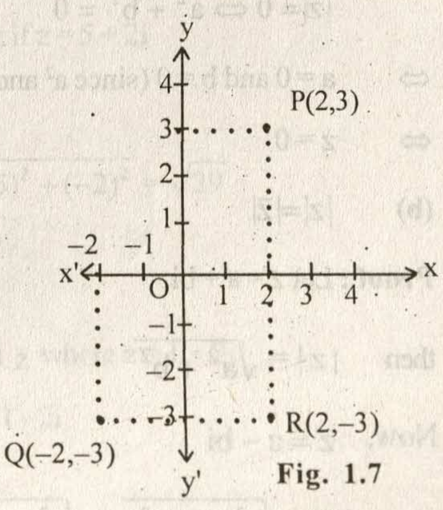

Fig. 1.7

1.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number z = a + bi can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing a + bi. Draw perpendiculars PM and PL on x-axis and y-axis respectively. Let OM = a and MP = b. We have to find the distance of P from the origin.

$$\therefore\ OP = \sqrt{OM^2 + MP^2}$$

$$= \sqrt{a^2 + b^2}$$

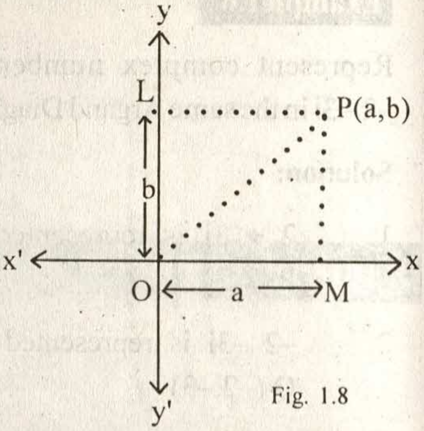

Fig. 1.8

OP is called the modulus or absolute value of the complex number a + bi.

$\therefore$ Modulus of any complex number z such that $z = a + bi$, $a \in R$, $b \in R$ is denoted by

$|z|$ and is given by $\sqrt{a^2 + b^2}$

$$\therefore \quad |z| = |a + ib| = \sqrt{a^2 + b^2}$$

1.5.1 Properties of Modulus

(a) $|z| = 0 \Leftrightarrow z = 0.$

Proof : Let $z = a + bi$, $a \in R$, $b \in R$

then $|z| = \sqrt{a^2 + b^2}$

$$|z| = 0 \Leftrightarrow a^2 + b^2 = 0$$

$\Leftrightarrow$ $a = 0$ and $b = 0$ (since a^2 and b^2 are both positive)

$\Leftrightarrow$ $z = 0$

(b) $|z| = |\overline{z}|$

Proof : Let $z = a + bi$

then $|z| = \sqrt{a^2 + b^2}$

Now, $\overline{z} = a - bi$

$$\therefore \quad |\overline{z}| = \sqrt{a^2 + (-b^2)} = \sqrt{a^2 + b^2}$$

Thus, $|z| = \sqrt{a^2 + b^2} = |\overline{z}|$ ...(i)

(c) $|z| = |-z|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

$$-z = -a - bi \text{ then } |-z| = \sqrt{(-a)^2 + (-b)^2}$$

$$= \sqrt{a^2 + b^2}$$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$ ...(ii)

By (i) and (ii) it can be proved that

$$|z| = |-z| = |\bar{z}| \qquad ...(iii)$$

Now, we consider the following examples:

Example 1.9 Find the modulus of z and $\bar{z}$ if $z = -4 + 3i$

Solution : $z = -4 + 3i$, then $|z| = \sqrt{(-4)^2 + (3)^2}$

$$= \sqrt{16 + 9} = \sqrt{25} = 5$$

and $\quad \bar{z} = -4 - 3i$

then, $\quad |\bar{z}| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

Thus, $\quad |z| = 5 = |\bar{z}|$

Example 1.10 Find the modulus of z and $-z$ if $z = 5 + 2i$

Solution: $z = 5 + 2i$, then $-z = -5 - 2i$

$$|z| = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ and } |-z| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

Thus, $\quad |z| = \sqrt{29} = |-z|$

Example 1.11 Find the modulus of z, $-z$ and $\bar{z}$ where $z = 1 + 2i$

Solution : $z = 1 + 2i$ then $-z = -1 - 2i$ and $\bar{z} = 1 - 2i$

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

and $\quad |\bar{z}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$

Thus, $\quad |z| = |-z| = \sqrt{5} = |\bar{z}|$

Example 1.12 Find the modulus of :

(i) $1+i$ (ii) 2π (iii) 0 (iv) $-\frac{1}{2}i$

Solution: (i) Let $z=1+i$

then $|z| = \sqrt{1^2+1^2} = \sqrt{2}$

Thus, $|1+i| = \sqrt{2}$

(ii) Let $z = 2\pi$ or $2\pi + 0i$

Then $|z| = \sqrt{(2\pi)^2 + (0)^2} = 2\pi$

Thus, $|2\pi| = 2\pi$.

(iii) $z = 0$ or $0 + 0i$

then $|z| = \sqrt{(0)^2 + (0)^2} = 0$

Thus, $|z|$

(iv) Let $z = -\frac{1}{2}i$ or $0 - \frac{1}{2}i$

then $|z| = \sqrt{0^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{2}$

Thus, $\left|-\frac{1}{2}i\right| = \frac{1}{2}$.

Example1.13 Find the absolute value of the conjugate of the complex number

$z = -2 + 3i$

Solution : Let $z = -2 + 3i$ then $\bar{z} = -2 - 3i$

Absolute value of $\bar{z} = |\bar{z}| = |-2 - 3i| =$

$$\sqrt{(-2)^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

Example 1.14 Find the modulus of the complex numbers shown in an Argand Plane (Fig. 1.9)

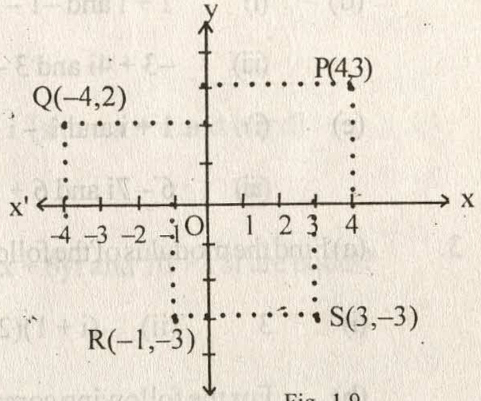

Fig. 1.9

Solution: (i) P(4, 3) represents the complex number $z = 4 + 3i$

$$\therefore \quad |z| = \sqrt{4^2 + 3^2} = \sqrt{25}$$

$$\text{or} \quad |z| = 5$$

(ii) Q(–4, 2) represents the complex number $z = -4 + 2i$

$$\therefore \quad |z| = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20}$$

$$\text{or} \quad |z| = 2\sqrt{5}$$

(iii) R(–1, –3) represents the complex number $z = -1 - 3i$

$$\therefore \quad |z| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1 + 9}$$

$$\text{or} \quad |z| = \sqrt{10}$$

(iv) S(3, –3) represents the complex number $z = 3 - 3i$

$$\therefore \quad |z| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9 + 9}$$

$$\text{or} \quad |z| = \sqrt{18} = 3\sqrt{2}$$

CHECK YOUR PROGRESS 1.2

1. Find the conjugate of each of the following:

(a) $-2i$ (b) $-5 - 3i$ (c) $-\sqrt{2}$ (d) $(-2 + i)^2$

2. Represent the following complex numbers on Argand Plane :

(a) (i) $2 + 0i$ (ii) $-3 + 0i$ (iii) $0 - 0i$ (iv) $3 - 0i$

(b) (i) $0+2i$ (ii) $0-3i$ (iii) $4i$ (iv) $-5i$

(c) (i) $2+5i$ and $5+2i$ (ii) $3-4i$ and $-4+3i$

(iii) $-7+2i$ and $2-7i$ (iv) $-2-9i$ and $-9-2i$

(d) (i) $1+i$ and $-1-i$ (ii) $6+5i$ and $-6-5i$

(iii) $-3+4i$ and $3-4i$ (iv) $4-i$ and $-4+i$

(e) (i) $1+i$ and $1-i$ (ii) $-3+4i$ and $-3-4i$

(iii) $6-7i$ and $6+7i$ (iv) $-5-i$ and $-5+i$

3. (a) Find the modulus of the following complex numbers :

(i) 3 (ii) $(i+1)(2-i)$ (iii) $2-3i$ (iv) $4+\sqrt{5}i$

(b) For the following complex numbers, verify that $|z|=|\bar{z}|$

(i) $-6+8i$ (ii) $-3-7i$

(c) For the following complex numbers, verify that $|z|=|-z|$

(i) $14+i$ (ii) $11-2i$

(d) For the following complex numbers, verify that $|z|=|-z|=|\bar{z}|$

(i) $2-3i$ (ii) $-6-i$ (iii) $7-2i$

1.6 EQUALITY OF TWO COMPLEX NUMBERS

Let us consider two complex numbers $z_1=a+bi$ and $z_2=c+di$ such that $z_1=z_2$

we have $a+bi=c+di$

or $(a-c)+(b-d)i=0=0+0i$

Comparing real and imaginary parts on both sides, we have

$a-c=0$, or $a=c$

$\Rightarrow$ real part of z_1 = real part of z_2

and $b-d=0$ or $b=d$

$\Rightarrow$ imaginary part of z_1 = imaginary part of z_2

Therefore, we can conclude that two complex numbers are equal if and only if their real parts and imaginary parts are equal.

In general $a+bi=c+di$ if and only if $a=c$ and $b=d$.

Properties; $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

Let $\quad z_1 = a + bi$ and $z_2 = c + di$

$z_1 = z_2$ gives $a = c$ and $b = d$

Now $\quad |z_1| = \sqrt{a^2 + b^2}$ and $|z_2| = \sqrt{c^2 + d^2}$

$$= \sqrt{a^2 + b^2} \quad \text{[since } a = c \text{ and } b = d\text{]}$$

$\Rightarrow \quad |z_1| = |z_2|$

Example 1.15 For what values of x and y, $5x + 6yi$ and $10 + 18i$ are equal?

Solution : It is given that

$$5x + 6y\,i = 10 + 18i$$

Comparing real and imaginary parts, we have

$5x = 10 \qquad$ or $x = 2$

and $\quad 6y = 18 \qquad$ or $y = 3$

For $x = 2$ and $y = 3$, the given complex numbers are equal.

1.7 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (a + c) + (b + d)i$$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$,

then $\quad z_1 + z_2 = [2 + (-4)] + [3 + 5]i$

$$= -2 + 8i.$$

Example 1.16 Simplify

(i) $\quad (3 + 2i) + (4 - 3i)$

(ii) $\quad (2 + 5i) + (-3 - 7i) + (1 - i)$

Solution : (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i = 7 - i$

(ii) $\quad (2 + 5i) + (-3 - 7i) + (1 - i) = (2 - 3 + 1) + (5 - 7 - 1)i$

$$= 0 - 3i$$

or $\quad (2 + 5i) + (-3 - 7i) + (1 - i) = -3i$

1.7.1 Geometrical Represention of Addition of Two Complex Numbers

Let two complex numbers z_1 and z_2 be represented by the points P(a, b) and Q(c, d).

Their sum, $z_1 + z_2$ is represented by the point R (a + c, b + d) in the same Argand Plane.

Join OP, OQ, OR, PR and QR.

Draw perpendiculars PM, QN, RL from P, Q, R respectively on x-axis.

Draw perpendicular PK to RL

In ΔQON

$ON = c$

and $QN = d$.

In ΔROL	In ΔPOM
$RL = b + d$	$PM = b$
and $OL = a + c$	$OM = a$

Also $PK = ML$

$= OL - OM$

$= a + c - a$

$= c = ON$

$RK = RL - KL$

$= RL - PM$

$= b + d - b$

$= d = QN$.

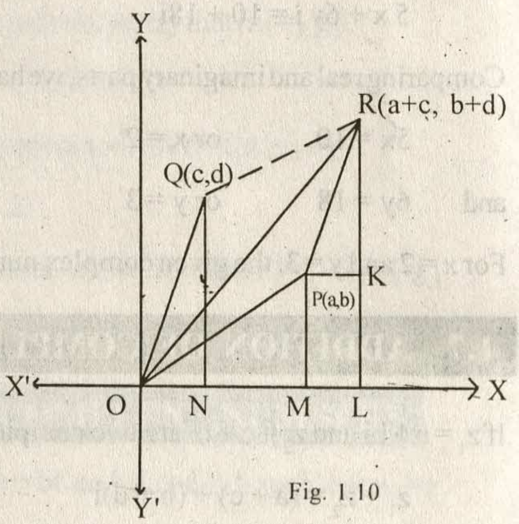

Fig. 1.10

In ΔQON and ΔRPK,

$ON = PK$, $QN = RK$ and $\angle QNO = \angle RKP = 90^\circ$

$\therefore \quad \Delta QON \cong \Delta RPK$

$\therefore \quad OQ = PR$ and $OQ \parallel PR$

$\Rightarrow$ OPRQ is a parallelogram and OR its diagonal.

Therefore, we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

Example 1.17 Prove that $|z_1 + z_2| \le |z_1| + |z_2|$

Solution: We have proved that the sum of two complex numbers z_1 and z_2 is represented by the diagonal of a parallelogram OPRQ (see fig. 1.11).

In ΔOPR

$$OR \le OP + PR$$

or $OR \le OP + OQ$ (since OQ=PR)

or $|z_1 + z_2| \le |z_1| + |z_2|$

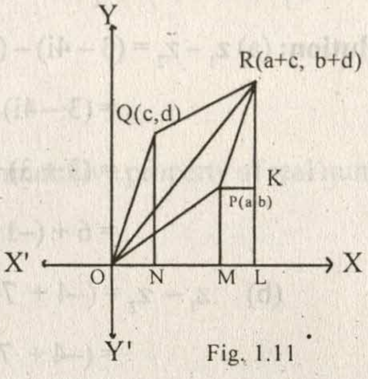

Fig. 1.11

Example 1.18 If $z_1 = 2 + 3i$ and $z_2 = 1 + i$,

verify that $|z_1 + z_2| \le |z_1| + |z_1|$

Solution: $z_1 = 2 + 3i$ and $z_2 = 1 + i$ are represented by the points (2, 3) and (1, 1) respectively. Their sum $(z_1 + z$;) will be represented by the point (2+1, 3+1) i.e. (3, 4)

Verification

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.6 \text{ approx.}$$

$$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41 \text{ approx.}$$

$$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|z_1| + |z_2| = 3.6 + 1.41 = 5.01$$

$\therefore$ $|z_1 + z_2| \le |z_1| + |z_2|$

1.7.2 Subtraction of the Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$\therefore$ $(z_1) - (z_2) = (a + bi) - (c + di)$

$= (a - c) + (b - d)i$

which represents a point $(a - c, b - d)$

$\therefore$ The difference i.e. $z_1 - z_2$ is represented by the point $(a - c, b - d)$.

Thus, to subtract a complex number from another, we subtract corresponding real and imaginary parts separately.

Example 1.19 Find $z_1 - z_2$ in each of following if:

(a) $z_1 = 3 - 4i, \quad z_2 = -3 + 7i$

(b) $z_1 = -4 + 7i \quad z_2 = -4 - 5i$

Solution: (a) $z_1 - z_2 = (3 - 4i) - (-3 + 7i)$

$$= (3 - 4i) + (3 - 7i)$$

$$= (3 + 3) + (-4 - 7)i$$

$$= 6 + (-11i) \quad = 6 - 11i$$

(b) $z_1 - z_2 = (-4 + 7i) - (-4 - 5i)$

$$= (-4 + 7i) + (4 + 5i)$$

$$= (-4 + 4) + (7 + 5)i$$

$$= 0 + 12i \ = 12i$$

Examle 1.20 What should be added to i to obtain 5+4*i*?

Solution: Let $z = a + bi$ be added to i to obtain $5 + 4i$

$\therefore \quad i + (a + bi) = 5 + 4i$

or, $\quad a + (b + 1)i = 5 + 4i$

Equating real and imaginary parts, we have

$a = 5$ and $b + 1 = 4$ or $b = 3$

$\therefore \quad z = 5 + 3i$ is to be added to i to obtain $5 + 4i$

1.8 PROPERTIES: WITH RESPECT TO ADDITION OF COMPLEX NUMBERS.

1. Closure : The sum of two complex numbers will always be a complex number.

Let $\quad z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i, \qquad a_1, b_1, a_2, b_2 \in R$.

Now, $\quad z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex number.

This proves the closure property of complex numbers.

Thus, $(1 + i) + (2 + 3i) = (1 + 2) + (1 + 3)i = 3 + 4i$, which is again a complex number.

Similarly, the difference of two complex numbers will always be a complex number. For example, $(2 + 4i) - (1 - 4i) = (2 - 1) + \{4 - (-4)\}i = 1 + 8i$, which is again a complex number.

2. Commutativity : If z_1 and z_2 are two complex numbers then

$$z_1 + z_2 = z_2 + z_1$$

Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$

Now $z_1 + z_2 = (a_1 + b_1i) + (a_2 + b_2i)$

$= (a_1 + a_2) + (b_1 + b_2)i$

$= (a_2 + a_1) + (b_2 + b_1)i$ [commutative property of real numbers]

$= (a_2 + b_2i) + (a_1 + b_1i)$

$= z_2 + z_1$

i.e. $z_1 + z_2 = z_2 + z_1$

Hence, addition of complex numbers is commutative.

For example, if $z_1 = 8 + 7i$ and $z_2 = 9 - 3i$ then

$z_1 + z_2 = (8 + 7i) + (9 - 3i)$ and $z_2 + z_1 = (9 - 3i) + (8 + 7i)$

$= (8 + 9) + (7 - 3)i$ and $= (9 + 8) + (-3 + 7)i$

or $z_1 + z_2 = 17 + 4i$ and $z_2 + z_1 = 17 + 4i$

We get, $z_1 + z_2 = z_2 + z_1$

Now, $z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$

$= (a_1 - a_2) + (b_1 - b_2)i$

and $z_2 - z_1 = (a_2 + b_2i) - (a_1 + b_1i)$

$= (a_2 - a_1) + (b_2 - b_1)i$

$= -(a_1 - a_2) - (b_1 - b_2)i$

$= -(a_1 + b_1i) + (a_2 + b_2i)$

$\therefore$ $z_1 - z_2 \neq z_2 - z_1$

Hence, subtraction of complex numbers is not commutative.

For example, if $z_1 = 8+7i$ and $z_2 = 9-3i$, then

$z_1 - z_2 = (8 + 7i) - (9 - 3i)$ and $z_2 - z_1 = (9 - 3i) - (8 + 7i)$

$= (8 - 9) + (7 + 3)i$ and $= (9 - 8) + (-3 - 7)i$

or $z_1 - z_2 = -1 + 10i$ and $z_2 - z_1 = 1 - 10i$

$z_1 - z_2 \neq z_2 - z_1$

3. Associativity

If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Now $z_1 + (z_2 + z_3)$

$$= (a_1 + b_1 i) + \{(a_2 + b_2 i) + (a_3 + b_3 i)\}$$

$$= (a_1 + b_1 i) + \{(a_2 + a_3) + (b_2 + b_3)i\}$$

$$= \{a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}i$$

$$= \{(a_1 + a_2) + (b_1 + b_2)i\} + (a_3 + b_3 i)$$

$$= \{(a_1 + b_1 i) + (a_2 + b_2 i)\} + (a_3 + b_3 i)$$

$$= (z_1 + z_2) + z_3$$

Hence, the associativity holds good in the case of addition of complex numbers.

For example, if $z_1 = 2 + 3i$, $z_2 = 3i$ and $z_3 = 1 - 2i$, then

$$z_1 + (z_2 + z_3) = (2 + 3i) + \{(3i) + (1 - 2i)\}$$

$$= (2 + 3i) + (1 + i)$$

$$= (3 + 4i)$$

and $(z_1 + z_2) + z_3 = \{(2 + 3i) + (3i)\} + (1 - 2i)$

$$= (2 + 6i) + (1 - 2i)$$

$$= (3 + 4i)$$

Thus, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

The equality of two sums is the consequence of the associative property of addition of complex numbers.

Like commutativity, it can be shown that associativity also does not hold good in the case of subtraction.

4. Existence of Additive Identitiy

If $x + yi$ be a complex number, then there exists a complex number $(0 + 0i)$ such that $(x + yi) + (0 + 0i) = x + yi$.

Let $z_2 = x + yi$ be the additive identity of $z_1 = 2 + 3i$ then

$$z_1 + z_2 = z_1$$

i.e. $(2 + 3i) + (x + yi) = 2 + 3i$

or $(2 + x) + (3 + y)i = 2 + 3i$

or $(2 + x) = 2$ and $3 + y = 3$

or $x = 0$ and $y = 0$

i.e. $z_2 = x + yi = 0 + 0i$ is the additive identity.

i.e. if $z = a + bi$ is any complex number, then

$(a + bi) + (0 + 0i) = a + bi$

i.e. $(0 + 0i)$ is the additive identity.

$$z_1 - z_2 = (2 + 3i) - (0 + 0i)$$
$$= (2 - 0) + (3 - 0)i$$
$$= 2 + 3i = z_1$$

∴ $z_2 = 0 + 0i$ is the identity w.r.t. subtraction also.

as $(a + bi) - (0 + 0i) = a + bi$

5. Existence of Additive Inverse

For every complex number $a + bi$ there exists a unique complex number $-a - bi$ such that $(a + bi) + (-a - bi) = 0 + 0i$

e.g. Let $z_1 = 4 + 5i$ and $z_2 = x + yi$ be the additive inverse of z_1

Then, $z_1 + z_2 = 0$

or $(4 + 5i) + (x + yi) = 0 + 0i$

or $(4 + x) + (5 + y)i = 0 + 0i$

or $4 + x = 0$ and $5 + y = 0$

or $x = -4$ and $y = -5$

Thus, $z_2 = -4 - 5i$ is the additive invese of $z_1 = 4 + 5i$

In general, additive inverse of a complex number is obtained by changing the signs of real and imaginary parts.

Consider $z_1 - z_2 = 0$

or $(4 + 5i) - (x + yi) = 0 + 0i$

or $(4 - x) + (5 - y)i = 0 + 0i$

or $4 - x = 0$ and $5 - y = 0$

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or $\quad x = 4 \quad$ and $y = 5$

i.e. $\quad z_1 - z_2 = 0$ gives $z_2 = 4 + 5i$

Thus, in subtraction, the number itself is the inverse.

i.e. $\quad (a + bi) - (a + bi) = 0 + 0i$ or 0

CHECK YOUR PROGRESS 1.3

1. Simplify:

(a) $\left(\sqrt{2} + \sqrt{5}i\right) + \left(\sqrt{5} - \sqrt{2}i\right)$ $\qquad$ (b) $\dfrac{2 + i}{3} + \dfrac{2 - i}{6}$

(c) $(1 + i) - (1 - 6i)$ $\qquad$ (d) $\left(\sqrt{2} - \sqrt{3}i\right) - (-2 - 7i)$

2. If $z_1 = (5 + i)$ and $z_2 = (6 + 2i)$, then:

(a) find $z_1 + z_2$ $\qquad$ (b) find $z_2 + z_1$ $\qquad$ (c) Is $z_1 + z_2 = z_2 + z_1$?

(d) find $z_1 - z_2$ $\qquad$ (e) find $z_2 - z_1$ $\qquad$ (f) Is $z_1 - z_2 = z_2 - z_1$?

3. If $z_1 = (1 + i)$, $z_2 = (1 - i)$ and $z_3 = (2 + 3i)$, then:

(a) find $z_1 + (z_2 + z_3)$ $\qquad$ (b) find $(z_1 + z_2) + z_3$

(c) Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? $\qquad$ (d) find $z_1 - (z_2 - z_3)$

(e) find $(z_1 - z_2) - z_3$ $\qquad$ (f) Is $z_1 - (z_2 - z_3) = (z_1 - z_2) - z_3$?

4. Find the additive inverse of the following:

(a) $12 - 7i$ $\qquad$ (b) $4 - 3i$

5. What shoud be added to $(-15 + 4i)$ to obtain $(3 - 2i)$?

6. Show that $\overline{\left\{(3 + 7i) - (5 + 2i)\right\}} = \overline{(3 + 7i)} - \overline{(5 + 2i)}$

1.9 ARGUMENT OF A COMPLEX NUMBER

Let P(a, b) represent the complex number $z = a + bi$, $a \in R$, $b \in R$, and OP makes an angle θ with the positive direction of x-axis.

Draw $PM \perp OX$

Let $OP = r$

In right Δ OMP

$OM = a$

$MP = b$

$\therefore \quad r\cos\theta = a$

$r\sin\theta = b$

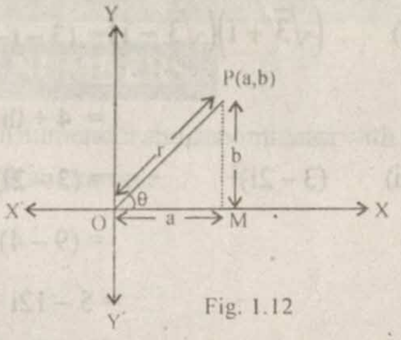

Fig. 1.12

Then $z = a + bi$ can be written as $z = r(\cos\theta + i\sin\theta)$ ...(i)

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$

or $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

This is known as the polar form of the complex number z, and r and θ are respectively called the modulus and argument of the complex number.

1.10 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of real numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then,

$z_1.z_2 = (a + bi).(c + di)$

$= a(c + di) + bi(c + di)$

or $= ac + adi + bci + bdi^2$

or $= (ac - bd) + (ad + bc)i.$ [since $i^2 = -1$]

If $(a + bi)$ and $(c + di)$ are two complex numbers, their product is defined as the complex number $(ac - bd) + (ad + bc)i$

Example 1.21 Evaluate:

(i) $(1 + 2i)(1 - 3i)$ (ii) $\left(\sqrt{3} + i\right)\left(\sqrt{3} - i\right)$ (iii) $(3 - 2i)^2$

Solution:

(i) $(1 + 2i)(1 - 3i) = \{1 - (-6)\} + (-3 + 2)i$

$= 7 - i$

(ii) $\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)=\{3-(-1)\}+\left(-\sqrt{3}+\sqrt{3}\right)i$

$$= 4+0i$$

(iii) $(3-2i)^2 \quad = (3-2i)\,(3-2i)$

$$= (9-4)+(-6-6)i$$

$$= 5-12i$$

1.10.1 Properties of Multiplication

$$|z_1.z_2| = |z_1|.|z_2|$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\therefore \quad |z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$.

Now, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\,r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]$$

$$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

[Since $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ and
$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$]

$$|z_1.z_2| = r_1 r_2\sqrt{\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)} = r_1 r_2$$

$$\therefore \quad |z_1.z_2| = r_1 r_2 = |z_1|.|z_2|$$

and argument of $z_1 z_2 = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Example 1.22 Find the modulus of the complex number $(1+i)(4-3i)$

Solution: Let $z = (1+i)(4-3i)$

then $|z| = |(1+i)(4-3i)|$

$$= |(1+i)|.|(4-3i)| \qquad (\text{since } |z_1 z_2| = |z_1|.|z_2|)$$

But $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$ and

$$|4-3i| = \sqrt{4^2+(-3)^2} = 5$$

$$\therefore \quad |z| = \sqrt{2}.5 = 5\sqrt{2}$$

1.11 DIVISION OF TWO COMPLEX NUMBERS

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator. We will explain it through an example.

Let $z_1 = a + bi$ and $z_2 = c+di$ then

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \qquad (c+di \neq 0)$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{(ac+bd)+(bc-ad)}{c^2+d^2}$$

Thus, $$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 1.23 Divide 3+i by 4–2i

Solution: $\dfrac{3+\mathrm{i}}{4-2\mathrm{i}}$

Multiplying numerator and denominator by the conjugate of (4 – 2i) we get

$$\frac{(3+i)(4+2i)}{(4-2i)(4+2i)}$$

$$= \frac{10+10i}{20}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Thus, $$\frac{3+\mathrm{i}}{4-2\mathrm{i}} = \frac{1}{2} + \frac{1}{2}\mathrm{i}$$

1.11.1 Properties of Division

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Proof: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$|z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$

and $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Thus, $$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}\sqrt{\cos^2(\theta_1 - \theta_2) + \sin^2(\theta_1 - \theta_2)} = \frac{r_1}{r_2}$$

$\therefore$ Argument of $$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

Example 1.24 Find the modulus of the complex number

$$\frac{2+i}{3-i}$$

Solution : Let $z = \dfrac{2+i}{3-i}$

$$\therefore \quad |z| = \left|\frac{2+i}{3-i}\right| = \frac{|2+i|}{|3-i|} \quad \left(\text{since } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$= \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-1)^2}} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad |z| = \frac{1}{\sqrt{2}}$$

1.12 PROPERTIES OF MULTIPLICATION OF TWO COMPLEX NUMBERS

1. Closure

If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then their product $z_1 z_2$ is also a complex number.

2. Cummutativity

If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then $z_1 z_2 = z_2 z_1$.

For example, let $z_1 = 3 + 4i$ and $z_2 = 1 - i$

then $\quad z_1 z_2 = (3 + 4i)(1 - i)$

$$= 3(1 - i) + 4i(1 - i)$$

$$= 3 - 3i + 4i - 4i^2$$

$$= 3 - 3i + 4i - 4(-1)$$

$$= 3 + i + 4 = 7 + i$$

Again, $\quad z_2 z_1 = (1 - i)(3 + 4i)$

$$= (3 + 4i) - i(3 + 4i)$$

$$= 3 + 4i - 3i - 4i^2$$

$$= 3 + i + 4 = 7 + i$$

$$\therefore \quad z_1 z_2 = z_2 z_1$$

3. Associativity

If $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$ then

$$z_1(z_2.z_3) = (z_1.z_2).z_3$$

Let us verify it with an example :

If $z_1 = (1+i), z_2 = (2+i)$ and $z_3 = (3+i)$ then

$$z_1(z_2.z_3) = (1+i)\{(2+i)(3+i)\}$$
$$= (1+i)\{(6-1)+(3+2)i\}$$
$$= (1+i)(5+5i)$$
$$= (5-5)+(5+5)i$$
$$= 0+10i = 10i$$

and $(z_1.z_2)z_3 = \{(1+i)(2+i)\}(3+i)$

$$= \{(2-1)+(1+2)i\}(3+i)$$
$$= (1+3i)(3+i)$$
$$= (3-3)+(1+9)i$$
$$= 0+10i = 10i$$

$\therefore \quad z_1(z_2.z_3) = (z_1.z_2)z_3$

4. Existence of Multiplicative Identity: For every non-zero complex number $z_1 = a+bi$ there exists a unique complex number $(1+0i)$ such that

$$(a+bi).(1+0i) = (1+0i)(a+bi) = a+bi$$

Let $z_1 = x+yi$ be the multipicative identity of $z = a+bi$

Then $z.z_1 = z.$

i.e. $(a+bi)(x+yi) = a+bi$

or $(ax-by)+(ay+bx)i = a+bi$

or $ax-by = a$ and $ay+bx = b$

or $x = 1$ and $y = 0$

i.e. $z_1 = x+yi = 1+0i$ is the multiplicative identity.

The complex number $1+0i$ is the identity for multiplication.

Let us verify it with an example:

If $z = 2+3i$ then

$$z.(1+0i) = (2+3i)(1+0i)$$
$$= (2-0)+(3+0)i$$
$$= 2+3i$$

5. Existence of Multiplicative inverse: Multiplicative inverse is a complex number that when multiplied to a given non-zero complex munber yields one. In other words, for every non-zero complex number z = a + bi, there exists a unique complex number (x + yi) such that their product is (1 + 0i).

i.e. $(a + bi)(x + yi) = 1 + 0i$

or $(ax - by) + (bx + ay)i = 1 + 0i$

Equating real and imaging parts, we have

$$ax - by = 1 \text{ and } bx + ay = 0$$

By cross multiplication

$$\frac{x}{a} = \frac{y}{-b} = \frac{1}{a^2 + b^2}$$

$$\Rightarrow \quad x = \frac{a}{a^2+b^2} = \frac{\operatorname{Re}(z)}{|z|^2} \text{ and } y = \frac{-b}{a^2+b^2} = -\frac{\operatorname{Im}(z)}{|z|^2}$$

Thus, the multiplicative inverse of a non-zero compelx number z = (a + bi) is

$$x + yi = \left(\frac{\operatorname{Re}(z)}{|z|^2} - \frac{\operatorname{Im}(z)}{|z|^2} i\right) = \frac{\bar{z}}{|z|^2}$$

Example 1.25 Find the multiplicative inverse of 2 – 4i.

Solution: Let $z = 2 - 4i$

We have, $\bar{z} = 2 + 4i$ and $|z|^2 = \left|2^2 + (-4)^2\right| = 20$

$\therefore$ Required multiplicative inverse is

$$\frac{\bar{z}}{|z|^2} = \frac{2+4i}{20} = \frac{1}{10} + \frac{1}{5}i$$

Verification:

If $\frac{1}{10} + \frac{1}{5}i$ be the muliplicative inverse of $2 - 4i$, there product must be equal to $1 + 0i$

We have, $(2 - 4i)\left(\frac{1}{10} + \frac{1}{5}i\right) = \left(\frac{2}{10} + \frac{4}{5}\right) + \left(\frac{2}{5} - \frac{4}{10}\right)i$

$= 1 + 0i$ which is true.

Notes

6. Distributive Property of Multiplication over Addition

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$

Then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Let us verify it with an example:

If $z_1 = 3 - 2i$, $z_2 = -1 + 4i$ and $z_3 = -3 - i$ then

$$\begin{aligned} z_1(z_2 + z_3) &= (3 - 2i)\{(-1 + 4i) + (-3 - i)\} \\ &= (3 - 2i)(-1 + 4i - 3 - i) \\ &= (3 - 2i)(-4 + 3i) \\ &= (-12 + 6) + (9 + 8)i \\ &= -6 + 17i \end{aligned}$$

and
$$\begin{aligned} z_1 z_2 &= (3 - 2i)(-1 + 4i) \\ &= (-3 + 8) + (12 + 2)i \\ &= 5 + 14i \end{aligned}$$

Again
$$\begin{aligned} z_1 z_3 &= (3 - 2i)(-3 - i) \\ &= (-9 - 2) + (-3 + 6)i \\ &= -11 + 3i \end{aligned}$$

Now
$$\begin{aligned} z_1 z_2 + z_1 z_3 &= (5 + 14i) + (-11 + 3i) \\ &= -6 + 17i \end{aligned}$$

$$\therefore \quad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

CHECK YOUR PROGRESS 1.4

1. Simplify each of the following:

(a) $(1 + 2i)(\sqrt{2} - i)$

(b) $(\sqrt{2} + i)^2$

(c) $(3 + i)(1 - i)(-1 + i)$

(d) $(2 + 3i) \div (1 - 2i)$

(e) $(1 + 2i) \div (1 + i)$

(f) $(1 + 0i) \div (3 + 7i)$

2. Compute multiplicative inverse of each of the following complex numbers:

(a) $3 - 4i$

(b) $\sqrt{3} + 7i$

(c) $\dfrac{3 + 5i}{2 - 3i}$

3. If $z_1 = 4 + 3i$, $z_2 = 3 - 2i$ and $z_3 = i + 5$, verify that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

4. If $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 2 - i$ then verify that $(z_1.z_2)z_3 = z_1(z_2.z_3)$

LET US SUM UP

- $z = a + bi$ is a complex number in the standard form where $a, b \in R$ and $i = \sqrt{-1}$.
- Any higher powers of 'i' can be expressed in terms of one of the four values $i, -1, -i, 1$.
- Conjugate of a complex number $z = a + bi$ is $a - bi$ and is denoted by $\bar{z}$.
- Modulus of a complex number $z = a + bi$ is $\sqrt{a^2 + b^2}$ i.e. $|z| = |a + bi| = \sqrt{a^2 + b^2}$

  (a) $|z| = 0 \Leftrightarrow z = 0$ (b) $|z| = |\bar{z}|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$

- $z = r(\cos\theta + i\sin\theta)$ represents the polar form of a complex number $z = a + bi$ where

  $r = \sqrt{a^2 + b^2}$ is modulus and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ is its argument.

- Multiplicative inverse of a complex number $z = a + bi$ is $\frac{\bar{z}}{|z|^2}$

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TERMINAL EXERCISE

1. Find real and imaginary parts of each of the following:

   (a) $2 + 7i$ (b) $3 + 0i$ (c) $-\frac{1}{2}$

   (d) $5i$ (e) $\frac{1}{2 + 3i}$

2. Simplify each of the following:

(a) $\sqrt{-3}.\sqrt{-27}$ (b) $\sqrt{-3}\sqrt{-4}\sqrt{-72}$ (c) $3i^{15}-5i^{8}+1$

3. Form complex number whose real and imaginary parts are given in the form of ordered pairs.

(a) $z(3,-5)$ (b) $z(0,-4)$ (c) $z(8,\pi)$

4. Find the conjugate of each of the following:

(a) $1-2i$ (b) $-1-2i$ (c) $6-\sqrt{2}i$

(d) $4i$ (e) $-4i$

5. Find the modulus of each of the following:

(a) $1-i$ (b) $3+\pi i$ (c) $-\frac{3}{2}i$ (d) $-2+\sqrt{3}i$

6. Express $7i^{17}-6i^{6}+3i^{3}-2i^{2}+1$ in the form of $a+bi$.

7. Find the values of x and y if:

(a) $(x-yi)+7-2i=9-i$

(b) $2x+3yi=4-9i$

(c) $x-3yi=7+9i$

8. Simplify each of the following:

(a) $(3+i)-(1-i)+(-1+i)$ (b) $\left(\frac{1}{7}+i\right)-\left(\frac{2}{7}-i\right)+\left(\frac{3}{7}-2i\right)$

9. Write additive inverse and multiplicative inverse of each of the following:

(a) $3-7i$ (b) $11-2i$ (c) $\sqrt{3}+2i$ (d) $1-\sqrt{2}i$ (e) $\frac{1+5i}{1-i}$

10. Find the modulus of each of the following complex numbers:

(a) $\frac{1+i}{3-i}$ (b) $\frac{5+2i}{\sqrt{2}+\sqrt{3}i}$

(c) $(3+2i)(1-i)$ (d) $(1-3i)(-2i^{3}+i^{2}+3)$

11. For the following pairs of complex numbers verify that $|z_1 z_2| = |z_2||z_1|$

    (a) $z_1 = 3 - 2i,\ z_2 = 1 - 5i$

    (b) $z_1 = 3 - \sqrt{7}i,\ z_2 = \sqrt{3} - i$

12. For the following pairs of complex numbers verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

    (a) $z_1 = 1 + 3i, \quad z_2 = 2 + 5i$

    (b) $z_1 = -2 + 5i, \quad z_2 = 3 - 4i$.

ANSWERS

CHECK YOUR PROGRESS 1.1

1 (a) $3\sqrt{3}i$ (b) $-3i$ (c) $\sqrt{13}i$

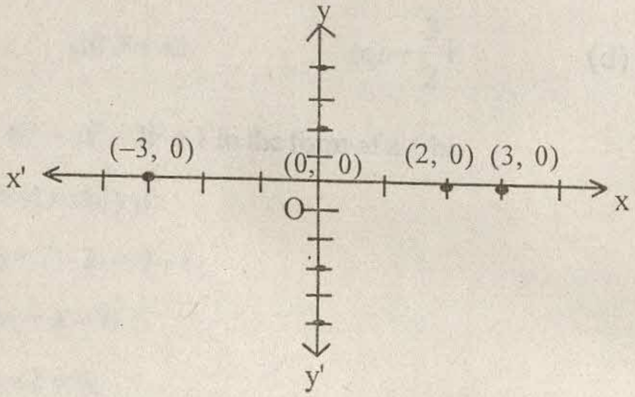
2. (a) $5+0i$ (b) $0-3i$ (c) $0+0i$

3. $12-4i$

CHECK YOUR PROGRESS 1.2

1 (a) $2i$ (b) $-5+3i$ (c) $-\sqrt{2}$ (d) $3+4i$

2. **(a)**



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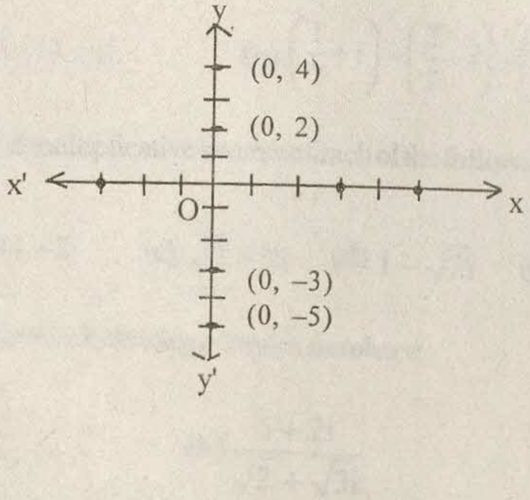

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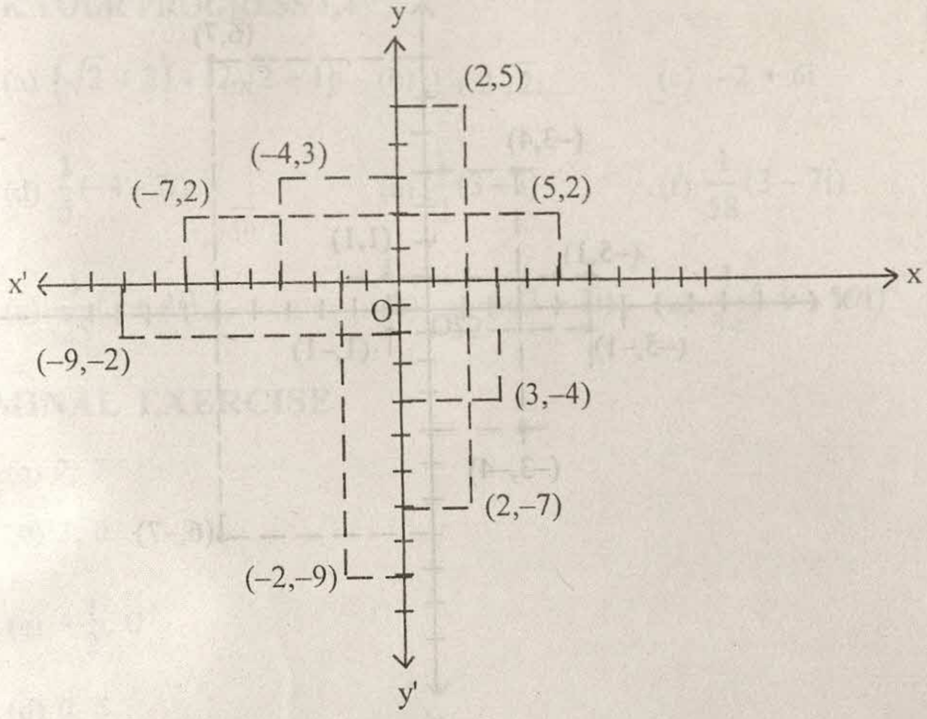

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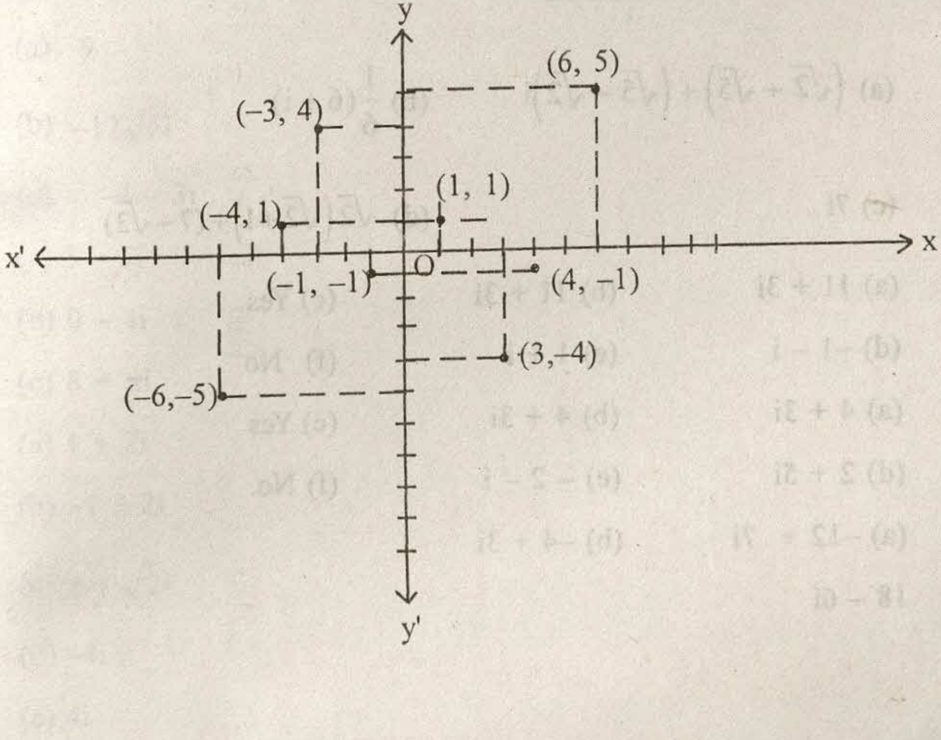

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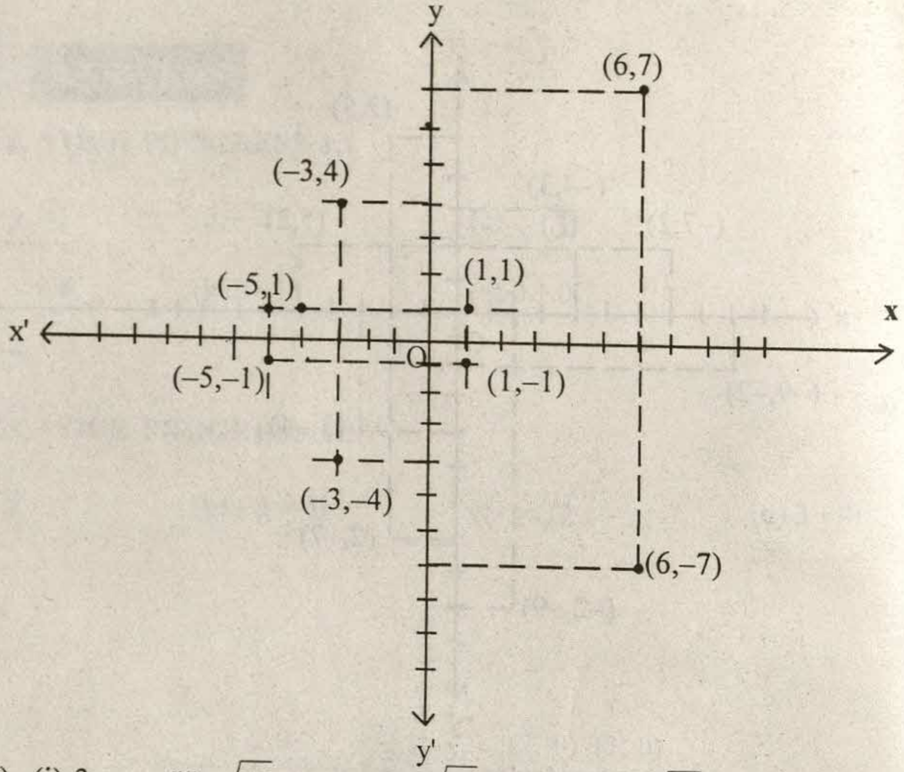


3. (a) (i) 3 (ii) $\sqrt{10}$ (iii) $\sqrt{13}$ (iv) $\sqrt{21}$

CHECK YOUR PROGRESS 1.3

1. (a) $\left(\sqrt{2}+\sqrt{5}\right)+\left(\sqrt{5}-\sqrt{2}\right)i$ (b) $\frac{1}{6}(6+i)$

(c) 7i (d) $\sqrt{2}\left(\sqrt{2}+1\right)+(7-\sqrt{3})$

2. (a) 11 + 3i (b) 11 + 3i (c) Yes

(d) –1 – i (e) 1 + i (f) No

3. (a) 4 + 3i (b) 4 + 3i (c) Yes

(d) 2 + 5i (e) – 2 – i (f) No.

4. (a) –12 + 7i (b) –4 + 3i

5. 18 – 6i

CHECK YOUR PROGRESS 1.4

1. (a) $\left(\sqrt{2}+2\right)+\left(2\sqrt{2}-1\right)i$ (b) $1+2\sqrt{2}i$ (c) $-2+6i$

   (d) $\frac{1}{5}(-4+7i)$ (e) $\frac{1}{2}(3+i)$ (f) $\frac{1}{58}(3-7i)$

2. (a) $\frac{1}{25}(3+4i)$ (b) $\frac{1}{52}(\sqrt{3}-7i)$ (c) $\frac{1}{34}(-9-19i)$

TERMINAL EXERCISE

1. (a) 2, 7
   (b) 3, 0
   (c) $-\frac{1}{2}$, 0
   (d) 0, 5
   (e) $\frac{2}{13}, -\frac{3}{13}$
2. (a) -9
   (b) $-12\sqrt{6}i$
   (c) $-4-3i$
3. (a) $3-5i$
   (b) $0-4i$
   (c) $8+\pi i$
4. (a) $1+2i$
   (b) $-1+2i$
   (c) $6+\sqrt{2}i$
   (d) $-4i$
   (e) $4i$

5. (a) $\sqrt{2}$

(b) $\sqrt{9+\pi^2}$

(c) $\frac{3}{2}$ (d) $\sqrt{7}$

6. 9 + 4i

7. (a) x = 2, y = –1

(b) x = 2, y = –3

(c) x = 7, y = – 3

8. (a) 1 + 3i

(b) $\frac{2}{7} + 0i$

9. (a) $-3 + 7i,\ \frac{1}{58}(3+7i)$

(b) $-11+2i,\ \frac{1}{125}(-11+2i)$

(c) $-\sqrt{3}-2i,\ \frac{1}{7}\left(\sqrt{3}-2i\right)$

(d) $-1+\sqrt{2}i,\ \frac{1}{3}(1+\sqrt{2}i)$

(e) $2-3i,\ \frac{1}{13}(2+3i)$

10. (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{5}\sqrt{145}$ (c) $\sqrt{26}$ (d) $4\sqrt{5}$

2

QUADRATIC EQUATIONS

Recall that an algebraic equation of the second degree is written in general form as

$ax^2 + bx + c = 0,\ a \neq 0$

It is called a quadratic equation in x. The coefficient 'a' is the first or leading coefficient, 'b' is the second or middle coefficient and 'c' is the constant term (or third coefficient).

For example, $7x^2 + 2x + 5 = 0$, $\frac{5}{2}x^2 + \frac{1}{2}x + 1 = 0$,

$3x^2 - x = 0$, $x^2 + \frac{1}{2} = 0$, $\sqrt{2}\,x^2 + 7x = 0$, are all quadratic equations.

In this lesson we will discuss how to solve quadratic equations with real and complex coefficients and establish relation between roots and coefficients. We will also find cube roots of unity and use these in solving problems.

OBJECTIVES

After studying this lesson, you will be able to:

- solve a quadratic equation with real coefficients by factorization and by using quadratic formula;
- find relationship between roots and coefficients;
- form a quadratic equation when roots are given; and
- find cube roots of unity.

EXPECTED BACKGROUND KNOWLEDGE

- Real numbers
- Quadratic Equations with real coefficients.

2.1 ROOTS OF A QUADRATIC EQUATION

The value which when substituted for the variable in an equation, satisfies it, is called a root (or solution) of the equation.

If α be one of the roots of the quadratic equation

then, $$ax^2+bx+c=0,\ a\neq 0 \qquad \text{...(i)}$$

$$a\alpha^2+b\alpha+c=0$$

In other words, $x-\alpha$ is a factor of the quadratic equation (i)

In particular, consider a quadratic equation $x^2+x-6=0$ ...(ii)

If we substitute $x=2$ in (ii), we get

$$\text{L.H.S} = 2^2+2-6=0$$

$$\therefore \quad \text{L.H.S} = \text{R.H.S.}$$

Again put $x=-3$ in (ii), we get

$$\text{L.H.S.} = (-3)^2-3-6=0$$

$$\therefore \quad \text{L.H.S} = \text{R.H.S.}$$

Again put $x=-1$ in (ii), we get

$$\text{L.H.S} = (-1)^2+(-1)-6=-6\neq 0=\text{R.H.S.}$$

$\therefore$ $x=2$ and $x=-3$ are the only values of x which satisfy the quadratic equation (ii)

There are no other values which satisfy (ii)

$\therefore$ $x=2, x=-3$ are the only two roots of the quadratic equation (ii)

Note: *If α, β be two roots of the quadratic equation*
$ax^2+bx+c=0,\ a\neq 0$ ...(A)
then $(x-\alpha)$ and $(x-\beta)$ will be the factors of (A). The given quadratic equation can be written in terms of these factors as $(x-\alpha)(x-\beta)=0$

2.2 SOLVING QUADRATIC EQUATION BY FACTORIZATION

Recall that you have learnt how to factorize quadratic polynomial of the form $p(x)=ax^2+bx+c,\ a\neq 0$, by splitting the middle term and taking the common factors. Same method can be applied while solving quadratic equation by factorization.

If $x-\frac{p}{q}$ and $x-\frac{r}{s}$ are two factors of the quadratic equation

$$ax^2+bx+c=0,\ a\neq 0 \text{ then } \left(x-\frac{p}{q}\right)\left(x-\frac{r}{s}\right)=0$$

$$\therefore \quad \text{either } x=\frac{p}{q} \text{ or, } x=\frac{r}{s}$$

$\therefore$ The roots of the quadratic equation $ax^2+bx+c=0$ are $\frac{p}{q}$, $\frac{r}{s}$

Example 2.1 Using factorization method, solve the following quadratic equation :

$$6x^2+5x-6=0$$

Solution: The given quadratic equation is

$$6x^2+5x-6=0 \qquad \text{...(i)}$$

Splitting the middle term, we have

$$6x^2+9x-4x-6=0$$

or, $3x(2x+3)-2(2x+3)=0$

or, $(2x+3)(3x-2)=0$

$\therefore$ Either $2x+3=0 \Rightarrow x=-\frac{3}{2}$

or, $3x-2=0 \Rightarrow x=\frac{2}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{2}, \frac{2}{3}$

Example 2.2 Using factorization method, solve the following quadratic equation:

$$3\sqrt{2}x^2+7x-3\sqrt{2}=0$$

Solution: Splitting the middle term, we have

$$3\sqrt{2}x^2+9x-2x-3\sqrt{2}=0$$

or, $3x(\sqrt{2}x+3)-\sqrt{2}(\sqrt{2}x+3)=0$

or, $(\sqrt{2}x+3)(3x-\sqrt{2})=0$

$\therefore$ Either $\sqrt{2}x+3=0 \Rightarrow x=-\frac{3}{\sqrt{2}}$

or, $3x-\sqrt{2}=0 \Rightarrow x=\frac{\sqrt{2}}{3}$

$\therefore$ Two roots of the given quadratic equation are $-\frac{3}{\sqrt{2}}, \frac{\sqrt{2}}{3}$

Example 2.3 Using factorization method, solve the following quadratic equation:

$$(a+b)^2 x^2 + 6(a^2 - b^2)x + 9(a-b)^2 = 0$$

Solution: The given quadratic equation is

$$(a+b)^2 x^2 + 6(a^2 - b^2)x + 9(a-b)^2 = 0$$

Splitting the middle term, we have

$$(a+b)^2 x^2 + 3(a^2 - b^2)x + 3(a^2 - b^2)x + 9(a-b)^2 = 0$$

or, $(a+b)x\{(a+b)x + 3(a-b)\} + 3(a-b)\{(a+b)x + 3(a-b)\} = 0$

or, $\{(a+b)x + 3(a-b)\}\{(a+b)x + 3(a-b)\} = 0$

$\therefore$ either $(a+b)x + 3(a-b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

or, $(a+b)x + 3(a-b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$.

The equal roots of the given quadratic equation are $\dfrac{3(b-a)}{a+b}, \dfrac{3(b-a)}{a+b}$

Alternative Method

The given quadratic equation is

$$(a+b)^2 x^2 + 6(a^2 - b^2)x + 9(a-b)^2 = 0$$

This can be rewritten as

$$\{(a+b)x\}^2 + 2.(a+b)x.3(a-b) + \{3(a-b)\}^2 = 0$$

or, $\{(a+b)x + 3(a-b)\}^2 = 0$

or, $x = -\dfrac{3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

$\therefore$ The quadratic equation have equal roots $\dfrac{3(b-a)}{a+b}, \dfrac{3(b-a)}{a+b}$

CHECK YOUR PROGRESS 2.1

1. Solve each of the following quadratic equations by factorization method:

(i) $\sqrt{3}\,x^2 + 10x + 8\sqrt{3} = 0$ (ii) $x^2 - 2ax + a^2 - b = 0$

(iii) $x^2 + \left(\dfrac{ab}{c} - \dfrac{c}{ab}\right)x - 1 = 0$ (iv) $x^2 - 4\sqrt{2}\,x + 6 = 0$

2.3 SOLVING QUADRATIC EQUATION BY QUADRATIC FORMULA

Recall the solution of a standard quadratic equation

$ax^2 + bx + c = 0, a \neq 0$ by the **"Method of Completing Squares"**

Roots of the above quadratic equation are given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and } x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{D}}{2a}, \qquad = \frac{-b-\sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is called the discriminant of the quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0, a \neq 0$ if

(i) D>0, the equation will have two real and unequal roots

(ii) D=0, the equation will have two real and equal roots and both roots are equal to $-\frac{b}{2a}$

(iii) D<0, the equation will have two conjugate complex (imaginary) roots.

Example 2.4 Examine the nature of roots in each of the following quadratic equations and also verify them by formula.

(i) $x^2 + 9x + 10 = 0$ (ii) $9y^2 - 6\sqrt{2}y + 2 = 0$

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

Solution:

(i) The given quadratic equation is $x^2 + 9x + 10 = 0$

Here, $a = 1, b = 9$ and $c = 10$

$\therefore$ $D = b^2 - 4ac = 81 - 4.1.10$

$= 41 > 0.$

$\therefore$ The equation will have two real and unequal roots

Verification: By quadratic formula, we have

$$x = \frac{-9 \pm \sqrt{41}}{2}$$

$\therefore$ The two roots are $\frac{-9+\sqrt{41}}{2}, \frac{-9-\sqrt{41}}{2}$ which are real and unequal.

(ii) The given quadratic equation is $9y^2 - 6\sqrt{2}\,y + 2 = 0$

Here, $D = b^2 - 4\,ac$

$$= (-6\sqrt{2})^2 - 4.9.2$$

$$= 72 - 72 = 0$$

$\therefore$ The equation will have two real and equal roots.

Verification: By quadratic formula, we have

$$y = \frac{6\sqrt{2} \pm \sqrt{0}}{2.9} = \frac{\sqrt{2}}{3}$$

$\therefore$ The two equal roots are $\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$.

(iii) The given quadratic equation is $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

Here, $D = (-3)^2 - 4.\sqrt{2}.3\sqrt{2}$

$$= -15 < 0$$

$\therefore$ The equation will have two conjugate complex roots.

Verification: By quadratic formula, we have

$$t = \frac{3 \pm \sqrt{-15}}{2\sqrt{2}}$$

$$= \frac{3 \pm \sqrt{15}i}{2\sqrt{2}}, \text{ where } i = \sqrt{-1}$$

$\therefore$ Two conjugate complex roots are $\frac{3+\sqrt{15}\,i}{2\sqrt{2}}, \frac{3-\sqrt{15}\,i}{2\sqrt{2}}$

Example 2.5 Prove that the quadratic equation $x^2 + py - 1 = 0$ has real and distinct roots for all real values of p.

Solution: Here, $D = p^2 + 4$ which is always positive for all real values of p.

$\therefore$ The quadratic equation will have real and distinct roots for all real values of p.

Example 2.6 For what value of k the quadratice equation

$$(4k+1)\,x^2 + (k+1)\,x + 1 = 0 \text{ will have equal roots ?}$$

Solution: The given quadratic equation is

$$(4k+1)x^2 + (k+1)\,x + 1 = 0$$

Here, $D = (k+1)^2 - 4.(4k+1).1$

For equal roots, $D = 0$

$\therefore \quad (k+1)^2 - 4\,(4k+1) = 0$

$\Rightarrow \quad k^2 - 14k - 3 = 0$

$$\therefore \quad k = \frac{14 \pm \sqrt{196+12}}{2}$$

$$\text{or} \quad k = \frac{14 \pm \sqrt{208}}{2}$$

$$= 7 \pm 2\sqrt{13} \text{ or } 7 + 2\sqrt{13},\, 7 - 2\sqrt{13}$$

which are the required values of k.

Example 2.7 Prove that the roots of the equation

$x^2\,(a^2+b^2) + 2x\,(ac+bd) + (c^2+d^2) = 0$ are imaginary. But if $ad = bc$, roots are real and equal.

Solution: The given equation is $x^2(a^2+b^2) + 2x\,(ac+bd) + (c^2+d^2) = 0$

$$\begin{aligned}\text{Discriminant} &= 4\,(ac+bd)^2 - 4\,(a^2+b^2)\,(c^2+d^2)\\ &= 8\,abcd - 4(a^2d^2 + b^2c^2)\\ &= -4\,(-2\,abcd + a^2d^2 + b^2c^2)\\ &= -4\,(ad - bc)^2\\ &< 0 \text{ for all a, b, c, d}\end{aligned}$$

$\therefore$ The roots of the given equation are imaginary.

For real and equal roots, discriminant is equal to zero.

$\Rightarrow \quad -4\,(ad - bc)^2 = 0$

or, $\quad ad = bc$

Hence, if ad=bc, the roots are real and equal.

CHECK YOUR PROGRESS 2.2

1. Solve each of the following quadratic equation by quadratic formula:

(i) $2x^2 - 3x + 3 = 0$ (ii) $-x^2 + \sqrt{2}x - 1 = 0$

(iii) $-4x^2 + \sqrt{5}x - 3 = 0$ (iv) $3x^2 + \sqrt{2}x + 5 = 0$

2. For what values of k will the equation

$y^2 - 2(1 + 2k)y + 3 + 2k = 0$ have equal roots ?

3. Show that the roots of the equation

$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are always real and they can not be equal unless $a = b = c$.

2.4 RELATION BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

You have learnt that, the roots of a quadratic equation $ax^2 + bx + c = 0, a \neq 0$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Let $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ ...(i) and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ ... (ii)

Adding (i) and (ii), we have

$$\alpha + \beta = \frac{-2b}{2a} = \frac{-b}{a}$$

$$\therefore \quad \textbf{Sum of the roots} = -\frac{\textit{coefficient of } x}{\textit{coefficient of } x^2} = -\frac{b}{a} \qquad \text{... (iii)}$$

$$\alpha\beta = \frac{+b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}$$

Notes

$\therefore$ **Product of the roots** $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{c}{a}$ ...(iv)

(iii) and (iv) are the required relationships between roots and coefficients of a given quadratic equation. These relationships helps to find out a quadratic equation when two roots are given.

Example 2.8 If, α, β are the roots of the equation $3x^2 - 5x + 9 = 0$ find the value of:

(a) $\alpha^2 + \beta^2$ (b) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution: (a) It is given that α, β are the roots of the quadratic equation $3x^2 - 5x + 9 = 0$.

$\therefore \quad \alpha + \beta = \dfrac{5}{3}$ ... (i)

and $\quad \alpha\beta = \dfrac{9}{3} = 3$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\dfrac{5}{3}\right)^2 - 2.3$ [By (i) and (ii)]

$= -\dfrac{29}{9}$

$\therefore$ The required value is $-\dfrac{29}{9}$

(b) Now, $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$= \dfrac{\dfrac{-29}{9}}{9}$ [By (i) and (ii)]

$= -\dfrac{29}{81}$

Example 2.9 If α, β are the roots of the equation $3y^2 + 4y + 1 = 0$, form a quadratic equation whose roots are α^2, β^2

Solution: It is given that α, β are two roots of the quadratic equation $3y^2 + 4y + 1 = 0$.

$\therefore$ Sum of the roots

i.e., $\alpha + \beta = -\dfrac{\text{coefficient of y}}{\text{coefficient of y}^2}$

$= -\dfrac{4}{3}$ ... (i)

Product of the roots i.e., $\alpha\ \beta = \dfrac{\text{constant term}}{\text{coefficient of y}^2}$

$= \dfrac{1}{3}$ ... (ii)

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\dfrac{4}{3}\right)^2 - 2.\dfrac{1}{3}$ [By (i) and (ii)]

$= \dfrac{16}{9} - \dfrac{2}{3}$

$= \dfrac{10}{9}$

and $\alpha^2\beta^2 = (\alpha\beta)^2 = \dfrac{1}{9}$ [By (ii)]

$\therefore$ The required quadratic equation is $y^2 - (\alpha^2 + \beta^2)y + \alpha^2\beta^2 = 0$

or, $y^2 - \dfrac{10}{9}y + \dfrac{1}{9} = 0$

or, $9y^2 - 10y + 1 = 0$

Example 2.10 If one root of the equation $ax^2 + bx + c = 0$, $a \neq 0$ be the square of the other, prove that $b^3 + ac^2 + a^2c = 3abc$

Solution: Let α, α^2 be two roots of the equation $ax^2 + bx + c = 0$.

$\therefore$ $\alpha + \alpha^2 = -\dfrac{b}{a}$ ... (i) and $\alpha \cdot \alpha^2 = \frac{c}{a}$

i.e., $\alpha^3 = \frac{c}{a}$ . ... (ii)

From (i) we have

$$\alpha\,(\alpha + 1) = -\frac{b}{a}$$

or, $\{\alpha\,(\alpha+1)\}^3 = \left(-\frac{b}{a}\right)^3 = -\frac{b^3}{a^3}$

or, $\alpha^3\,(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = -\frac{b^3}{a^3}$

or, $\frac{c}{a}\left\{\frac{c}{a} + 3\left(-\frac{b}{a}\right) + 1\right\} = -\frac{b^3}{a^3}$ ... [By (i) and (ii)]

or, $\frac{c^2}{a^2} - \frac{3bc}{a^2} + \frac{c}{a} = -\frac{b^3}{a^3}$

or, $ac^2 - 3abc + a^2c = -b^3$

or, $b^3 + ac^2 + a^2c = 3abc$

which is the required result.

Example 2.11 Find the condition that the roots of the equation $ax^2 + bx + c = 0$ are in the ratio m : n

Solution: Let $m\alpha$ and $n\alpha$ be the roots of the equation $ax^2 + bx + c = 0$

Now, $m\alpha + n\alpha = -\frac{b}{a}$ ... (i)

and $mn\alpha^2 = \frac{c}{a}$ ... (ii)

From (i) we have, $\alpha\,(m+n) = -\frac{b}{a}$

or, $\alpha^2\,(m+n)^2 = \frac{b^2}{a^2}$

or, $\frac{c}{a}(m+n)^2 = mn\,\frac{b^2}{a^2}$ [By (ii)]

or, $ac\,(m+n)^2 = mn\,b^2$

which is the required condition

CHECK YOUR PROGRESS 2,3

1. If α, β are the roots of the equation $ay^2 + by + c = 0$ then find the value of :

   (i) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (ii) $\frac{1}{\alpha^4} + \frac{1}{\beta^4}$

2. If α, β are the roots of the equation $5x^2 - 6x + 3 = 0$, form a quadratic equation whose roots are:

   (i) α^2, β^2 (ii) $\alpha^3\beta, \alpha\beta^3$

3. If the roots of the equation $ay^2 + by + c = 0$ be in the ratio 3:4, prove that $12b^2 = 49\,ac$

4. Find the condition that one root of the quadratic equation $px^2 - qx + p = 0$ may be 1 more than the other.

2,5 SOLUTION OF A QUADRATIC EQUATION WHEN D < 0

Let us consider the following quadratic equation:

(a) Solve for t :

$t^2 + 3t + 4 = 0$

$$\therefore \quad t = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Here, $D = -7 < 0$

$\therefore$ The roots are $\frac{-3+\sqrt{-7}}{2}$ and $\frac{-3-\sqrt{-7}}{2}$

or, $\frac{-3+\sqrt{7}\,i}{2}, \frac{-3-\sqrt{7}\,i}{2}$

Thus, the roots are complex and conjugate.

(b) Solve for y :

$$-3y^2+\sqrt{5}\,y-2=0$$

$$\therefore \quad y=\frac{-\sqrt{5}\pm\sqrt{5-4(-3).(-2)}}{2(-3)}$$

or $$y=\frac{-\sqrt{5}\pm\sqrt{-19}}{-6}$$

Here, $D=-19<0$

$\therefore$ The roots are $\frac{-\sqrt{5}+\sqrt{19}i}{-6},\frac{-\sqrt{5}-\sqrt{19}i}{-6}$

Here, also roots are complex and conjugate. From the above examples , we can make the following conclusions:

(i) $D<0$ in both the cases

(ii) Roots are complex and conjugate to each other.

Is it always true that complex roots occur in conjugate pairs ?

Let us form a quadratic equation whose roots are

$2+3i$ and $4-5i$

The equation will be $\{x-(2+3i)\}\ \{x-(4-5i)\}=0$

or, $x^2-(2+3i)x-(4-5i)x+(2+3i)\,(4-5i)=0$

or, $x^2+(-6+2i)x+23+2i=0$

which is an equation with complex coefficients.

> ***Note :*** *If the quadratic equation has two complex roots, which are not conjugate of each other, the quadratic equation is an equation with complex coefficients.*

2.6 CUBE ROOTS OF UNITY

Let us consider an equation of degree 3 or more. Any equation of degree 3 can be expressed as a product of a linear equaion and a quadratic equation.

The simplest situation that comes for our consideration is

$$x^3-1=0 \qquad \text{...(i)}$$

$$\therefore \quad x^3-1=(x-1)\,(x^2+x+1)=0$$

either $x-1=0$ or, $x^2+x+1=0$

or, $x = 1$ or, $x = \dfrac{-1 \pm \sqrt{3}i}{2}$.

$\therefore$ Roots are $1, -\dfrac{1}{2} + \dfrac{\sqrt{3}\,i}{2}$ and $-\dfrac{1}{2} - \dfrac{\sqrt{3}\,i}{2}$

These are called **cube roots of unity.**

Do you see any relationship between two non-real roots of unity obtained above ?

Let us try to find the relationship between them

Let $w = -\dfrac{1}{2} + \dfrac{\sqrt{3}\,i}{2}$

Squaring both sides, we have

$$w^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right)^2$$

$$= \frac{1}{4}(1 + 3i^2 - 2\sqrt{3}\,i)$$

$$= \frac{1}{4}(1 - 3 - 2\sqrt{3}\,i)$$

$$= \frac{-2 - 2\sqrt{3}\,i}{4}$$

$$= \frac{-2(1 + \sqrt{3}\,i)}{4} = \frac{-(1 + \sqrt{3}\,i)}{2}$$

$\therefore$ $w^2 = -\dfrac{1}{2} - \dfrac{\sqrt{3}\,i}{2}$ = other complex root.

$\therefore$ We denote cube roots of unity as $1, w, w^2$

Note: If we take $w = -\frac{1}{2} - \frac{\sqrt{3}i}{2}$, we can verify, that in this case

also $w^2 = -\frac{1}{2} + \frac{\sqrt{3}\,i}{2}$ = other complex root.

Thus, square of one complex root is same as the other complex root.

Again, sum of roots i.e., $1 + w + w^2$

$$= \quad 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}\,i}{2}\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}\,i}{2}\right)$$

$$= \quad 1 - \frac{1}{2} - \frac{1}{2}$$

$$= \quad 1 - 1$$

$$= \quad 0$$

$$\therefore \quad 1 + w + w^2 = 0$$

Sum of cube roots of unity is zero.

Product of cube roots of 1

i.e., $1.w.w^2 = w^3 = 1$ (since ,w is a root the equation $x^3 - 1 = 0$)

We can conclude that , if 1, w and w^2 are cube roots of unity then

(i) square of one complex root is same as the other complex root i.e., $w^2 = w$

(ii) $1 + w + w^2 = 0$

(iii) $w^3 = 1$

Let us now consider another equation

$$x^3 + 1 = 0 \qquad \text{... (ii)}$$

or $\quad (x + 1)(x^2 - x + 1) = 0$

Either, $\quad x + 1 = 0 \quad$ or, $\quad x^2 - x + 1 = 0$

or, $\quad x = -1 \quad$ or, $\quad x = \frac{1 \pm \sqrt{1-4}}{2}$

$\Rightarrow \quad x = -1 \quad$ or, $\quad x = \frac{1 \pm \sqrt{-3}}{2}$

$\therefore$ Roots are $-1, \frac{1}{2}-\frac{\sqrt{3}\,i}{2}$ and $\frac{1}{2}+\frac{\sqrt{3}\,i}{2}$ which can also be written as -1, $-w$ and $-w^2$

Therefore, cube roots of -1 are -1, $-w$ and $-w^2$

In general, roots of any cubic equation of the form $x^3 = \pm a^3$ would be $\pm a$, $\pm aw$ and $\pm aw^2$

Example 2.12 If 1, w and w^2 are cube roots of unity, prove that

(a) $1 + w^2 + w^7 = 0$

(b) $(1 - w + w^2)(1 + w - w^2) = 4$

(c) $(1 + w)^3 - (1 + w^2)^3 = 0$

(d) $(1 - w + w^2)^3 = -8$ and $(1 + w - w^2)^3 = -8$

Solution:

(a) $1 + w^2 + w^7 = 0$

$$\begin{aligned}
\text{L.H.S} &= 1 + w^2 + (w^3)^2 . w \\
&= 1 + w^2 + w && [\text{since } w^3 = 1] \\
&= 0 && [\text{since } 1 + w + w^2 = 0] \\
&= \text{R.H.S}
\end{aligned}$$

$\therefore$ L.H.S = R.H.S

(b) $(1 - w + w^2)(1 + w - w^2) = 4$

$$\begin{aligned}
\text{L.H.S} &= (1 - w + w^2)(1 + w - w^2) \\
&= (1 + w^2 - w)(1 + w - w^2) \\
&\quad (\text{since } 1 + w^2 = -w \text{ and } \quad 1 + w = -w^2) \\
&= (-w - w)(-w^2 - w^2) \\
&= (-2w)(-2w^2) \\
&= 4w^3 \\
&= 4.1 = 4 = \text{R.H.S}
\end{aligned}$$

$\therefore$ L.H.S = R.H.S

(c) $(1+w)^3 - (1+w^2)^3 = 0$

L.H.S $= (1+w)^3 - (1+w^2)^3$

$= (-w^2)^3 - (-w)^3$ $(\therefore\ 1+w = -w^2$

$= -w^6 + w^3$ and $1+w^2 = -w)$

$= -(w^3)^2 + 1$

$= -(1)^2 + 1$

$= 0 =$ R.H.S

$\therefore$ L.H.S = R.H.S

(d) $(1 - w + w^2)^3 = -8$ and $(1 + w - w^2)^3 = -8$

Case I : L.H.S $= (1 - w + w^2)^3$

$= (1 + w^2 - w)^3$

$= (-w - w)^3$

$= (-2w)^3$

$= -8w^3$

$= -8 =$ R.H.S

$\therefore$ L.H.S = R.H.S

Case II : L.H.S $= (1 + w - w^2)^3$

$= (-w^2 - w^2)^3$

$= (-2w^2)^3$

$= -8w^6$

$= -8(w^3)^2$

$= -8 =$ R.H.S

$\therefore$ L.H.S = R.H.S

CHECK YOUR PROGRESS 2.4

1. Solve each of the following cubic equations:

(i) $x^3 = 27$ (ii) $x^3 = -27$

(iii) $x^3 = 64$ (iv) $x^3 = -64$

2. If 1, w, w^2 are cube roots of unity, show that

(i) $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8) = 1$

(ii) $(1 - w)(1 - w^2)(1 - w^4)(1 - w^5) = 9$

(iii) $(1 + w)^4 + (1 + w^2)^4 = -1$

(iv) $(1 + w^3)^3 = 8$

(v) $(1 - w + w^2)^6 = (1 + w - w^2)^6 = 64$

(vi) $(1 + w)^{16} + w = (1 + w^2)^{16} + w^2 = -1$

LET US SUM UP

- Roots of the quadratic equation $ax^2 + bx + c = 0$ are complex and conjugate of each other, when $D < 0$.
- If α, β be the roots of the quadratic equation

$ax^2 + bx + c = 0$ then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

- Cube roots of unity are 1, w, w^2

where $w = -\dfrac{1}{2} + \dfrac{\sqrt{3}\,i}{2}$ and $w^2 = -\dfrac{1}{2} - \dfrac{\sqrt{3}\,i}{2}$

- Sum of cube roots of unity is zero i.e., $1 + w + w^2 = 0$
- Product of cube roots of unity is 1 i.e., $w^3 = 1$
- Complex roots w and w^2 are conjugate to each other.
- In general roots of any cubic equation of the form

$x^3 = \pm a^3$ would be $\pm a$, $\pm aw$ and $\pm aw^2$

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Show that the roots of the equation

   $2(a^2+b^2)x^2+2(a+b)x+1=0$ are imaginary, when $a \neq b$

2. Show that the roots of the equation

   $bx^2+(b-c)x=c+a-b$ are always real if those of

   $ax^2+b(2x+1)=0$ are imaginary.

3. If α, β be the roots of the equation $2x^2-6x+5=0$, find the equation whose roots are:

   (i) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ (ii) $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$ (iii) $\alpha^2+\beta^2, \dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

4. If 1, w and w^2 are cube roots of unity, prove that

   (a) $(2-w)(2-w^2)(2-w^{10})(2-w^{11})=49$

   (b) $(x-y)(xw-y))(xw^2-y)=x^3-y^3$

5. If $x=a+b$, $y=aw+bw^2$ and $z=aw^2+bw$, then prove that

   (a) $x^2+y^2+z^2=6ab$ (b) $xyz=a^3+b^3$

ANSWERS

CHECK YOUR PROGRESS 2.1

1. (i) $-2\sqrt{3}, \dfrac{-4}{\sqrt{3}}$ (ii) $a-\sqrt{b}, a+\sqrt{b}$

(iii) $-\dfrac{ab}{c}, \dfrac{c}{ab}$ (iv) $3\sqrt{2}, \sqrt{2}$

CHECK YOUR PROGRESS 2.2

1. (i) $\dfrac{3 \pm \sqrt{15}\,i}{4}$ (ii) $\dfrac{1 \pm i}{\sqrt{2}}$

(iii) $\dfrac{\sqrt{5} \pm \sqrt{43}\,i}{8}$ (iv) $\dfrac{-\sqrt{2} \pm \sqrt{58}\,i}{6}$

2. $-1, \dfrac{1}{2}$

CHECK YOUR PROGRESS 2.3

1. (i) $\dfrac{b^2-2ac}{c^2}$ (ii) $\dfrac{(b^2-2ac)^2-2a^2c^2}{c^4}$

2. (i) $25x^2-6x+9=0$ (ii) $625x^2-90x+81=0$

4. $q^2-5p^2=0$

CHECK YOUR PROGRESS 2.4

1. (i) $3, 3w, 3w^2$ (ii) $-3, -3w, -3w^2$

(iii) $4, 4w, 4w^2$ (iv) $-4, -4w, -4w^2$

TERMINAL EXERCISE

3. (i) $5x^2-8x+5=0$ (ii) $10x^2-42x+49=0$ (iii) $25x^2-116x+64=0$

3

MATRICES

In the middle of the 19th Century, Arthur Cayley (1821-1895), an English mathematician created a new discipline of mathematics, called **matrices**. He used matrices to represent simultaneous system of equations. As of now, theory of matrices has come to stay as an important area of mathematics. The matrices are used in game theory, allocation of expenses, budgeting for by-products etc. Economists use them in social accounting, input-output tables and in the study of inter-industry economics. Matrices are extensively used in solving the simultaneous system of equations. Linear programming has its base in matrix algebra. Matrices have found applications not only in mathematics, but in other subjects like Physics, Chemistry, Engineering, Linear Programming etc.

In this lesson we will discuss different types of matrices and algebraic operations on matrices in details.

OBJECTIVES

After studying this lesson, you will be able to:

- define a matrix, order of a matrix and cite examples thereof;
- define and cite examples of various types of matrices-square, rectangular, unit, zero, diagonal, row, column matrix;
- state the conditions for equality of two matrices;
- define transpose of a matrix;
- define symmetric and skew symmetric matrices and cite examples;
- find the sum and the difference of two matrices of the same order;
- multiply a matrix by a scalar;
- state the condition for multiplication of two matrices; and
- multiply two matrices whenever possible.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number system
- Solution of system of linear equations

3.1 MATRICES AND THEIR REPRESENTATIONS

Suppose we wish to express that Anil has 6 pencils. We may express it as [6] or (6) with the understanding that the number inside [ ] denotes the number of pencils that Anil has. Next suppose that we want to express that Anil has 2 books and 5 pencils. We may express it as [2 5] with the understanding that the first entry inside [ ] denotes the number of books; while the second entry, the number of pencils, possessed by Anil.

Let us now consider, the case of two friends Shyam and Irfan. Shyam has 2 books, 4 notebooks and 2 pens; and Irfan has 3 books, 5 notebooks and 3 pens.

A convenient way of representing this information is in the tabular form as follows:

	Books	Notebooks	Pens
Shyam	2	4	2
Irfan	3	5	3

We can also briefly write this as follows:

$$\begin{array}{rc} & \begin{array}{ccc}\text{First Column} & \text{Second Column} & \text{Third Column}\\ \downarrow & \downarrow & \downarrow\end{array} \\ \begin{array}{r}\text{First Row}\\ \text{Second Row}\end{array} & \left[\begin{array}{ccc} 2 & 4 & 2\\ 3 & 5 & 3\end{array}\right]\end{array}$$

This representation gives the following information:

(1) The entries in the first and second rows represent the number of objects (Books, Notebooks, Pens) possessed by Shyam and Irfan, respectively

(2) The entries in the first, second and third columns represent the number of books, the number of notebooks and the number of pens, respectively.

Thus, the entry in the first row and third column represents the number of pens possessed by Shyam. Each entry in the above display can be interpreted similarly.

The above information can also be represented as

	Shyam	Irfan
Books	2	3
Notebooks	4	5
Pens	2	3

which can be expressed in three rows and two columns as given below:

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$$

The arrangement is called a **matrix.** Usually, we denote a matrix by a capital letter of English alphabets, i.e. A, B, X, etc. Thus, to represent the above information in the form of a matrix, we write

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}$$

Note: Plural of matrix is matrices.

3.1.1 Order of a Matrix

Observe the following matrices (arrangement of numbers):

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & i \\ i & 1+i \\ 1+i & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & 3 & 4 & 5 \\ 4 & -1 & -2 & 0 \end{bmatrix}$

In matrix (a), there are two rows and two columns, this is called a 2 by 2 matrix or a matrix of order 2×2. This is written as 2×2 matrix. In matrix (b), there are three rows and two columns. It is a 3 by 2 matrix or a matrix of order 3×2. It is written as 3×2 matrix. The matrix (c) is a matrix of order 3×4.

Note that there may be any number of rows and any number of columns in a matrix. If there are m rows and n columns in matrix A, its order is $m \times n$ and it is read as an $m \times n$ matrix.

Use of two suffixes i and j helps in locating any particular element of a matrix. In the above $m \times n$ matrix, the element a_{ij} belongs to the ith row and jth column.

$$A = \begin{bmatrix} a_{11}\, a_{12}\, a_{13} \cdots a_{1j} & \cdots a_{1n} \\ a_{21}\, a_{22}\, a_{23} \cdots a_{2j} & \cdots a_{2n} \\ a_{31}\, a_{32}\, a_{33} \cdots a_{3j} & \cdots a_{3n} \\ a_{i1}\, a_{i2}\, a_{i3} \cdots\, a_{ij} & \cdots a_{in} \\ a_{m1}\, a_{m2}\, a_{m3} \cdots a_{mj} & \cdots a_{mn} \end{bmatrix}.$$

A matrix of order $m \times n$ can also be written as

$A = [a_{ij}]$, $i = 1, 2, ..., m$; and
$j = 1, 2, ..., n$

Example 3.1 Write the order of each of the following matrices:

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{bmatrix}$

Solution: The order of the matrix

(i) is 2×2 (ii) is 3×1
(iii) is 1×3 (iv) is 2×3

Example 3.2 For the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & 6 \end{bmatrix}.$$

(i) find the order of A
(ii) write the total number of elements in A
(iii) write the elements a_{23}, a_{32}, a_{14} and a_{34} of A
(iv) express each element 3 in A in the form a_{ij}.

Solution:

(i) Since A has 3 rows and 4 columns, A is of order 3×4.
(ii) number of elements in $A = 3 \times 4 = 12$
(iii) $a_{23} = 2$; $a_{32} = 2$; $a_{14} = 4$ and $a_{34} = 6$
(iv) a_{22}, a_{31} and a_{33}

Example 3.3 If the element in the ith row and jth column of a 2 × 3 matrix A is given by $\frac{i+2j}{2}$, write the matrix A.

Solution: Here, $a_{ij} = \frac{i+2j}{2}$ (Given)

$$a_{11} = \frac{1+2\times 1}{2} = \frac{3}{2};\ a_{12} = \frac{1+2\times 2}{2} = \frac{5}{2};\ a_{13} = \frac{1+2\times 3}{2} = \frac{7}{2}$$

$$a_{21} = \frac{2+2\times 1}{2} = 2;\ a_{22} = \frac{2+2\times 2}{2} = 3;\ a_{23} = \frac{2+2\times 3}{2} = 4$$

Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 \end{bmatrix}$$

Example 3.4 There are two stores A and B. In store A, there are 120 shirts, 100 trousers and 50 cardigans; and in store B, there are 200 shirts, 150 trousers and 100 cardigans. Express this information in tabular form in two different ways and also in the matrix form.

Solution:

Tabular Form 1

	Shirts	Trousers	Cardigans
Store A	120	100	50
Store B	200	150	100

Matrix Form

$$\Rightarrow \begin{bmatrix} 120 & 100 & 50 \\ 200 & 150 & 100 \end{bmatrix}$$

Tabular Form 2

	Store A	Store B
Shirts	120	200
Trousers	100	150
Cardigans	50	100

Matrix Form

$$\Rightarrow \begin{bmatrix} 120 & 200 \\ 100 & 150 \\ 50 & 100 \end{bmatrix}$$

CHECK YOUR PROGRESS 3.1

1. Marks scored by two students A and B in three tests are given in the adjacent table. Represent this information in the matrix form, in two ways

	Test 1	Test 2	Test 3
A	56	65	71
B	29	37	57

2. Three firms X, Y and Z supply 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively, to a contractor. Express this information in the matrix form in two ways.

3. In family P, there are 4 men, 6 women and 3 children; and in family Q, there are 4 men, 3 women and 5 children. Express this information in the form of a matrix of order 2×3.

4. How many elements are there in a
(a) 2×3 matrix? (b) 3×4 matrix? (c) 4×2 matrix?
(d) 6×2 matrix? (e) $a \times b$ matrix? (f) $m \times n$ matrix?

5. What are the possible orders of a matrix if it has
(a) 8 elements (b) 5 elements (c) 12 elements (d) 16 elements

6. In the matrix A,

$$A = \begin{bmatrix} 5 & 1 & 8 & 0 & 5 \\ 7 & 6 & 7 & 4 & 6 \\ 3 & 9 & 3 & -3 & 9 \\ 4 & 4 & 8 & 5 & 1 \end{bmatrix}$$

find:

(a) number of rows;
(b) number of columns;
(c) the order of the matrix A;
(d) the total number of elements in the matrix A;
(e) elements $a_{14}, a_{23}, a_{34}, a_{45}$ and a_{33}

7. Construct a 3×3 matrix whose elements in the ith row and jth column is given by

(a) $i - j$ (b) $\frac{i^2}{j}$ (c) $\frac{(i+2j)^2}{2}$ (d) $3j - 2i$

8. Construct a 3×2 matrix whose elements in the ith row and jth column is given by
(a) $i + 3j$ (b) $5.i.j.$ (c) i^j (d) $i + j - 2$

3.2 TYPES OF MATRICES

Row Matrix : A matrix is said to be a *row matrix* if it has only one row. It may have any number of columns, e.g. the matrix $\begin{bmatrix} 1 & 6 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix. The order of a row matrix is $1 \times n$.

Column Matrix : A matrix is said to be a *column matrix* if it has only one column, but may have any number of rows, e.g. the matrix $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}$ is a column matrix.

The order of a column matrix is $m \times 1$

Square Matrix : A matrix is said to be a *square matrix* if number of rows is equal to the number of columns, e.g. the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ having 3 rows and 3 columns is a square matrix.

The order of a square matrix is $n \times n$ or simply n.

The diagonal of a square matrix from the top extreme left element to the bottom extreme right element is said to be the principal diagonal. The principal diagonal of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ contains elements 2, 1 and 9.

Note: In any given matrix $A = [a_{ij}]$ of order $m \times n$, the elements of the principal diagonal are $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$

Rectangular Matrix : A matrix is said to be a *rectangular matrix* if the number of rows is not equal to the number of columns, e.g. the matrix

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$$

having 3 rows and 4 columns is a rectangular matrix.It may be noted that a row matrix of order $1 \times n \ (n \neq 1)$ and a column matrix of order $m \times 1 \ (m \neq 1)$ are rectangular matrices.

Zero or Null Matrix : A matrix each of whose element is zero is called a *zero or null matrix,* e.g. each of the matrices

Notes

$$[0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix. Zero matrix is denoted by O.

Note: A zero matrix may be of any order $m \times n$.

Diagonal Matrix : A square matrix is said to be a *diagonal matrix,* if all elements other than those occuring in the principal diagonal are zero, i.e., if $A = [a_{ij}]$ is a square matrix of order n × n, then it is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$ are diagonal matrices.

Note: A diagonal matrix $A = [a_{ij}]_{n \times n}$ is also written as $A =$ diag $[a_{11}, a_{12}, a_{13}, \ldots, a_{nn}]$

Scalar Matrix : A diagonal matrix is said to be a *scalar matrix* if all the elements in its principal diagonal are equal to some non-zero constant, say k e.g., the matrix $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix.

Note: A square zero matrix is not a scalar matrix.

Unit or Identity Matrix : A scalar matrix is said to be a *unit or identity matrix*, if all of its elements in the principal diagonal are unity. It is denoted by I_n, if it is of order n e.g., the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: A square matrix $A = [a_{ij}]$ is a unit matrix if

$$a_{ij} = \begin{cases} 0, when\, i \neq j \\ 1, when\, i = j \end{cases}$$

Equal Matrices : Two matrices are said to be equal if they are of the same order and if their corresponding elements are equal.

If A is a matrix of order $m \times n$ and B is a matrix of order $p \times r$, then

$A = B$ if

(1) $m = p$; $n = r$; and

(2) $a_{ij} = b_{ij}$ for all $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$

Two matrices X and Y given below are not equal, since they are of different orders, namely 2×3 and 3×2 respectively.

$$X = \begin{bmatrix} 7 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}, \; Y = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

Also, the two matrices P and Q are not equal, since some elements of P are not equal to the corresponding elements of Q.

$$P = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix}, \; Q = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

Example 3.5 Find whether the following matrices are equal or not:

(i) $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \end{bmatrix}, \; Q = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}$

Solution: (i) Matrices A and B are of the same order 2×2. But some of their corresponding elements are different. Hence, $A \neq B$.

(ii) Matrices P and Q are of different orders, So, $P \neq Q$.

(iii) Matrices X and Y are of the same order 3×3, and their corresponding elements are also equal.

So, $X = Y$.

Example 3.6 Determine the values of x and y, if

(i) $\begin{bmatrix} x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \end{bmatrix}$

(iii) $\begin{bmatrix} x & 2 \\ 3 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Since the two matrices are equal, their corresponding elements should be equal.

(i) $x = 2$
(ii) $x = 4, \ y = 3$
(iii) $x = 1, \ y = -5$

Example 3.7 For what values of a, b, c and d, are the following matrices equal?

(i) $A = \begin{bmatrix} a & -2 & 2b \\ 6 & 3 & d \end{bmatrix}, \ B = \begin{bmatrix} 1 & -2 & 4 \\ 6 & 5c & 2 \end{bmatrix}$

(ii) $P = \begin{bmatrix} a & b-2d \\ -3 & 2b \\ a+c & 7 \end{bmatrix}, \ Q = \begin{bmatrix} 5 & 1 \\ -3 & 6 \\ 4 & 7 \end{bmatrix}$

Solution: (i) The given matrices A and B will be equal only if their corresponding elements are equal, i.e. if

$$a=1, \ 2b=4, \ 3=5c, \ \text{and } d=2$$

$$\Rightarrow a=1, \ b=2, \ c=\frac{3}{5} \ \text{and } d=2$$

Thus, for $a = 1$, $b = 2$, $c = \frac{3}{5}$ and $d = 2$ matrices A and B are equal.

(ii) The given matrices P and Q will be equal if their corresponding elements are equal, i.e. if

$$a = 5,\ b - 2d = 1,\ 2b = 6 \text{ and } a + c = 4$$

$$\Rightarrow a = 5,\ b = 3,\ c = -1 \text{ and } d = 1$$

Thus, for $a = 5,\ b = 3,\ c = -1$ and $d = 1$, matrices P and Q are equal.

CHECK YOUR PROGRESS 3.2

1. Which of the following matrices are

   (a) row matrices (b) column matrices (c) square matrices (d) diagonal matrices (e) scalar matrices (f) identity matrices and (g) zero matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix},\ B = \begin{bmatrix} 2 \\ 7 \\ 8 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 9 & 8 \\ 1 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 3 & 4 & 10 & 8 \end{bmatrix},\ H = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 4 & 9 \end{bmatrix},\ I = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$$

2. Find the values of a, b, c and d if

   (a) $\begin{bmatrix} b & 2c \\ b+d & c-2a \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 8 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} a+2 & 4 \\ b+3 & 25 \end{bmatrix} = \begin{bmatrix} 4 & 2c \\ 6 & 5d \end{bmatrix}$

   (c) $\begin{bmatrix} 2a & b \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ d & 3c \end{bmatrix}$

3. Can a matrix of order 1 × 2 be equal to a matrix of order 2 × 1?

4. Can a matrix of order 2 × 3 be equal to a matrix of order 3 × 3?

3.3 TRANSPOSE OF A MATRIX

Associated with each given matrix there exists another matrix called its *transpose*. The transpose of a given matrix A is formed by interchanging its rows and columns and is denoted by A' or A^t, e.g. if

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 6 & 1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \\ -3 & 3 & 1 \end{bmatrix}$$

In general, If $A = [a_{ij}]$ is an $m \times n$ matrix, then the transpose A' of A is the $n \times m$ matrix; and, (a_{ij})th element of $A = (a_{ji})$th element of A'

3.3.1 Symmetric Matrix

A square matrix A is said to be a *symmetric matrix* if $A' = A$.

For example,

$$If\ A = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 2 & 3i & 1-i \\ 3i & 4 & 2i \\ 1-i & 2i & 5 \end{bmatrix}$$

Since $A' = A$, A is a symmetric matrix.

Note: (1) In a symmetric matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = a_{ji}$ for all i and j

(2) A rectangular matrix can never be symmetric.

3.3.2 Skew-Symmetric Matrix

A square matrix A is said to be a *skew symmetric* if $A' = -A$, i.e. $a_{ij} = -a_{ji}$ for all i and j.

For example,

$$\text{If } A = \begin{bmatrix} 0 & c & d \\ -c & 0 & f \\ -d & -f & 0 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 0 & -c & -d \\ c & 0 & -f \\ d & f & 0 \end{bmatrix}$$

But $-A=\begin{bmatrix}0 & -c & -d\\ c & 0 & -f\\ d & f & 0\end{bmatrix}$, which is the same as A'

$$A'=-A$$

Hence, A is a skew symmetric matrix

Note: In a skew symmetric matrix $A=\left[a_{ij}\right]_{n\times n}$, $a_{ij}=0$, for $i=j$

i.e. all elements in the principal diagonal of a skew symmetric matrix are zeroes.

3.4 SCALAR MULTIPLICATION OF A MATRIX

Let us consider the following situation:

The marks obtained by three students in English, Hindi, and Mathematics are as follows:

	English	Hindi	Mathematics
Elizabeth	20	10	15
Usha	22	25	27
Shabnam	17	25	21

It is also given that these marks are out of 30 in each case. In matrix form, the above information can be written as

$$\begin{bmatrix}20 & 10 & 15\\ 22 & 25 & 27\\ 17 & 25 & 21\end{bmatrix}$$

(It is understood that rows correspond to the names and columns correspond to the subjects)

If the maximum marks are doubled in each case, then the marks obtained by these girls will also be doubled. In matrix form, the new marks can be given as:

$$\begin{bmatrix}2\times 20 & 2\times 10 & 2\times 15\\ 2\times 22 & 2\times 25 & 2\times 27\\ 2\times 17 & 2\times 25 & 2\times 21\end{bmatrix} \text{ which is equal to } \begin{bmatrix}40 & 20 & 30\\ 44 & 50 & 54\\ 34 & 50 & 42\end{bmatrix}$$

So, we write that

$$2\times\begin{bmatrix}20 & 10 & 15\\ 22 & 25 & 27\\ 17 & 25 & 21\end{bmatrix}=\begin{bmatrix}2\times 20 & 2\times 10 & 2\times 15\\ 2\times 22 & 2\times 25 & 2\times 27\\ 2\times 17 & 2\times 25 & 2\times 21\end{bmatrix}=\begin{bmatrix}40 & 20 & 30\\ 44 & 50 & 54\\ 34 & 50 & 42\end{bmatrix}$$

Now consider another matrix

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix}$$

Let us see what happens, when we multiply the matrix A by 5

$$\text{i.e.} \quad 5 \times A = 5A = 5 \times \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 \times 3 & 5 \times 2 \\ 5 \times (-2) & 5 \times 0 \\ 5 \times 1 & 5 \times 6 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ -10 & 0 \\ 5 & 30 \end{bmatrix}$$

When *a* matrix is multiplied by a scalar, then each of its element is multiplied by the same scalar.

For example,

$$\text{if } A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix} \text{ then } kA = \begin{bmatrix} k \times 2 & k \times (-1) \\ k \times 6 & k \times 3 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ 6k & 3k \end{bmatrix}$$

When $k = -1$, $kA = (-1)A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

So, $(-1)A = -A$

Thus, if $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$

Example 3.8 If $A = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, find .

(i) $2A$ (ii) $\frac{1}{2}A$ (iii) $-A$ (iv) $\frac{2}{3}A$

Solution:

(i) Here, $2A = 2 \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times (-2) & 2 \times 3 & 2 \times 4 \\ 2 \times (-1) & 2 \times 0 & 2 \times 1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 8 \\ -2 & 0 & 2 \end{bmatrix}$

(ii) $$\frac{1}{2}A = \frac{1}{2} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times (-2) & \frac{1}{2} \times 3 & \frac{1}{2} \times 4 \\ \frac{1}{2} \times (-1) & \frac{1}{2} \times 0 & \frac{1}{2} \times 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(iii) $$-A = (-1) \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -1 \end{bmatrix}$$

iv) $$\frac{2}{3}A = \frac{2}{3} \times \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

CHECK YOUR PROGRESS 3.3

1. If $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, find:

(a) $4A$ (b) $-A$ (c) $\frac{1}{2}A$ (d) $-\frac{3}{2}A$

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find:

(a) $5A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $-\frac{1}{2}A$

3. If $A = \begin{bmatrix} -1 & 0 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$, find $(-7)A$

4. If $X = \begin{bmatrix} 3 & 0 & 1 \\ 4 & -2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$, find:

(a) $5X$ (b) $-4X$ (c) $\frac{1}{3}X$ (d) $-\frac{1}{2}X$

5. Find A' (transpose of A):

(a) $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & 10 & 9 \\ 6 & 8 & 7 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -2 \\ 4 & -1 \\ -6 & 9 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. For any matrix A, prove that $(A')' = A$

7. Show that each of the following matrices is a symmetric matrix:

(a) $\begin{bmatrix} 2 & -4 \\ -4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Show that each of the following matrices is a skew symmetric matrix:

(a) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & i & 4 \\ -i & 0 & 2-i \\ -4 & -2+i & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix}$

3.5 ADDITION OF MATRICES

Two students A and B compare their performances in two tests in Mathematics, Physics and English. The maximum marks in each test in each subject are 50. The marks scored by them are as follows:

First Test

$$\begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 50 & 38 & 33 \\ 47 & 40 & 36 \end{bmatrix} \end{array}$$

Second Test

$$\begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 45 & 32 & 30 \\ 42 & 30 & 39 \end{bmatrix} \end{array}$$

How can we find their total marks in each subject in the two tests taken together ?

Observe that the new matrix giving the combined information of two matrices

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 50+45 & 38+32 & 33+30 \\ 47+42 & 40+30 & 36+39 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{ccc} M & P & E \end{array} \\ \begin{bmatrix} 95 & 70 & 63 \\ 89 & 70 & 75 \end{bmatrix} \end{array}$$

This new matrix is called the *sum* of the given matrices.

If A and B are any two given matrices of the same order, then their sum is defined to be a matrix C whose respective elements are the sum of the corresponding elements of the matrices A and B and we write this as $C = A + B$.

Notes:

1. The order of the matrix C will also be the same as that of A and B.
2. It is not possible to add two matrices of different orders.

Example 3.9 If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 1 & 0 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A + B = \begin{bmatrix} 1+5 & 3+2 \\ 4+1 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 \\ 5 & 2 \end{bmatrix}$$

Example 3.10 If $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \end{bmatrix}$, then find $A + B$.

Solution: Since the given matrices A and B are of the same order, i.e. 2×3, we can add them. So,

$$A + B = \begin{bmatrix} 0+3 & 1+0 & -1+4 \\ 2+1 & 3+2 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

3.5.1 Properties of Additio

Recall that in case of numbers, we have

(i) $x + y = y + x$, i.e., addition is commutative

(ii) $x + (y + z) = (x + y) + z$, i.e., addition is associative

(iii) $x + 0 = x$, i.e., additive identity exists

(iv) $x + (-x) = 0$, i.e., additive inverse exists

Let us now find if these properties hold true in case of matrices too:

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$, Then,

$$A + B = \begin{bmatrix} 1+0 & 2-2 \\ -1+1 & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$B + A = \begin{bmatrix} 0+1 & -2+2 \\ 1+(-1) & 3+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

We see that $A + B$ and $B + A$ denote the same matrix. Thus, in general,

For any two matrices A and B of the same order, $A + B = B + A$

i.e. matrix addition is commutative

Let $A = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$. Then,

$$A + (B + C) = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1+1 & -4+0 \\ 0+2 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 3+(-4) \\ -2+2 & 1+5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$$

and $$(A+B)+C=\begin{bmatrix}0+1 & 3+(-4)\\ -2+0 & 1+2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1 & -1+0\\ -2+2 & 3+3\end{bmatrix}$$

$$=\begin{bmatrix}2 & -1\\ 0 & 6\end{bmatrix}$$

We see that $A+(B+C)$ and $(A+B)+C$ denote the same matrix. Thus, in general

For any three matrices A, B and C of the same order,
$A+(B+C)=(A+B)+C$ i.e., matrix addition is associative.

Recall that we have talked about zero matrix. A zero matrix is that matrix, all of whose elements are zeroes. It can be of any order.

Let $A=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ Then,

$$A+O=\begin{bmatrix}2+0 & -2+0\\ 4+0 & 5+0\end{bmatrix}=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}=A$$

and $$O+A=\begin{bmatrix}0+2 & 0-2\\ 0+4 & 0+5\end{bmatrix}=\begin{bmatrix}2 & -2\\ 4 & 5\end{bmatrix}=A$$

We see that $A+O$ and $O+A$ denote the same matrix A.

Thus, we find that $A+O=A=O+A$, where O is a zero matrix.

The matrix O, which is a zero matrix, is called the additive identity.

Additive identity is a zero matrix, which when added to a given matrix, gives the same given matrix, i.e., $A+O=A=O+A$.

Example 3.11 If $A=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, then find: (a) $A+B$ (b) $B+C$ (c) $(A+B)+C$ (d) $A+(B+C)$

Solution:

(a) $$A+B=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2+(-3) & 0+1\\ 1+1 & 3+2\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}$$

(b) $$B+C=\begin{bmatrix}-3 & 1\\ 1 & 2\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}(-3)+(-1) & 1+0\\ 1+0 & 2+3\end{bmatrix}=\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix}$$

(c) $$(A+B)+C=\begin{bmatrix}-1 & 1\\ 2 & 5\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix} \quad \text{... [From (a)]}$$

$$=\begin{bmatrix}(-1)+(-1) & 1+0\\ 2+0 & 5+3\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$$

(d) $$A+(B+C)=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+\begin{bmatrix}-4 & 1\\ 1 & 5\end{bmatrix} \quad \text{... [From (b)]}$$

$$=\begin{bmatrix}2+(-4) & 0+1\\ 1+1 & 3+5\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 2 & 8\end{bmatrix}$$

Example 3.12 If $A=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

then find (a) $A+O$ (b) $O+A$

What do you observe?

Solution: (a) $$A+O=\begin{bmatrix}-2 & 3 & 5\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix} -2+0 & 3+0 & 5+0 \\ 1+0 & -1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

(b) $$O + A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+(-2) & 0+3 & 0+5 \\ 0+1 & 0+(-1) & 0+0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

From (a) and (b), we see that

$$A + O = O + A = A$$

3.6 SUBTRACTION OF MATRICES

Let A and B two matrices of the same order. Then the matrix A–B is defined as the subtraction of B from A. A–B is obtained by subtracting corresponding elements of B from the corresponding elements of A.
We can write $A - B = A + (-B)$

Note : $A-B$ and $B-A$ do not denote the same matrix, except when $A = B$.

Example 3.13 If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find

(a) $A-B$ (b) $B-A$

Solution : (a) We know that

$$A - B = A + (-B) \qquad \text{(i)}$$

Since $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, we have $-B = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$

Substituting it in (i), we get

$$A - B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+(-2) \\ 2+(-1) & (-1)+(-4) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

(b) Similarly,

$$B - A = B + (-A)$$

$$B-A=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}+\begin{bmatrix}-1 & 0\\ -2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3+(-1) & 2+0\\ 1+(-2) & 4+1\end{bmatrix}=\begin{bmatrix}2 & 2\\ -1 & 5\end{bmatrix}$$

Remarks : To obtain $A-B$, we can subtract directly the elements of B from the corresponding elements of A. Thus,

$$A-B=\begin{bmatrix}1-3 & 0-2\\ 2-1 & -1-4\end{bmatrix}=\begin{bmatrix}-2 & -2\\ 1 & -5\end{bmatrix}$$

and $$B-A=\begin{bmatrix}3-1 & 2-0\\ 1-2 & 4-(-1)\end{bmatrix}=\begin{bmatrix}2 & 2\\ -1 & 5\end{bmatrix}$$

Example 3.14 Find (a) $A-B$ (b) $B-A$, for the matrices A and B defined as under:

$$A=\begin{bmatrix}2 & 4\\ 5 & 7\\ -1 & 4\end{bmatrix},\ B=\begin{bmatrix}0 & 3\\ 5 & -2\\ 3 & 1\end{bmatrix}$$

Solution : (a) $$A-B=\begin{bmatrix}2 & 4\\ 5 & 7\\ -1 & 4\end{bmatrix}-\begin{bmatrix}0 & 3\\ 5 & -2\\ 3 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2-0 & 4-3\\ 5-5 & 7-(-2)\\ -1-3 & 4-1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 0 & 9\\ -4 & 3\end{bmatrix}$$

(b) $$B-A=\begin{bmatrix}0 & 3\\ 5 & -2\\ 3 & 1\end{bmatrix}-\begin{bmatrix}2 & 4\\ 5 & 7\\ -1 & 4\end{bmatrix}$$

$$=\begin{bmatrix}0-2 & 3-4\\ 5-5 & -2-7\\ 3-(-1) & 1-4\end{bmatrix}=\begin{bmatrix}-2 & -1\\ 0 & -9\\ 4 & -3\end{bmatrix}$$

Remarks : From above examples we can conclude that the matrix subtraction is not commutative.

Example 3.15 If $A=\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}$; $B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $A+B=O$, find B.

Solution : Here, it is given that $A+B=O$

$$\therefore \quad \begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}+\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2+a & 3+b\\ -1+c & 4+d\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \quad 2+a=0 \quad ; \quad 3+b=0$$
$$-1+c=0 \quad ; \quad 4+d=0$$

$$\Rightarrow \quad a=-2;\ b=-3; \quad c=1 \text{ and } d=-4$$

$$\therefore \quad B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}-2 & -3\\ 1 & -2\end{bmatrix}$$

Remarks : In Example 3.15 the elements of B are the additive inverse of the corresponding elements of A. We, therefore, call B, the additive inverse of the matrix A. Further,

$$B=\begin{bmatrix}-2 & -3\\ 1 & -4\end{bmatrix}=(-1)\times\begin{bmatrix}2 & 3\\ -1 & 4\end{bmatrix}=(-1)\times A=-A$$

In general, given a matrix A, there exists another matrix $B=(-1)A$ such that $A+B=O$, then such a matrix B is called the additive inverse of the matrix of A.

CHECK YOUR PROGRESS 3.4

1. If $A=\begin{bmatrix}3 & -1\\ 5 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\ 3 & 2\end{bmatrix}$ then find :

   (a) $A+B$ (b) $2A+B$ (c) $A+3B$ (d) $2A+3B$

2. If $P=\begin{bmatrix}2 & 5 & 3\\ -1 & 4 & 0\end{bmatrix}$ and $Q=\begin{bmatrix}1 & 2 & -3\\ 4 & 1 & -5\end{bmatrix}$, then find :

   (a) P–Q (b) Q–P (c) P–2Q (d) 2Q–3P

3. If $A=\begin{bmatrix}1 & -2 & 3\\ 4 & -1 & 2\\ 4 & 5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}-1 & -4 & 0\\ 1 & 6 & 1\\ 2 & 0 & 7\end{bmatrix}$, then find:

(a) $A+B$ (b) $A-B$ (c) $-A+B$ (d) $3A+2B$

4. If $A=\begin{bmatrix}0 & 1\\ 0 & -1\\ -1 & 1\end{bmatrix}$, find the zero matrix O satisfying A+O = A.

5. If $A=\begin{bmatrix}-2 & -1 & 0\\ 1 & 2 & 3\\ -4 & 0 & 1\end{bmatrix}$ then find :

(a) −A (b) A + (−A) (c) (−A)+A

6. If $A=\begin{bmatrix}1 & 9\\ 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}5 & 1\\ 7 & 9\end{bmatrix}$, then find :

(a) 2A (b) 3B (c) 2A+3B (d) If 2A+3B+5X=O, what is X ?

7. If $A=\begin{bmatrix}2 & 0 & -1\\ 4 & 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 0 & 1\\ 2 & -4 & 0\end{bmatrix}$, then find :

(a) A′ (b) B′ (c) A+B (d) (A+B)′ (e) A′+B′

What do you observe ?

8. If $A=\begin{bmatrix}1 & 4\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$ and $C=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$, then find :

(a) A−B (b) B−C (c) A−C (d) 3B−2C (e) A−B−C (f) 2A−B−3C

3.7 MULTIPLICATION OF MATRICES

Salina and Rakhi are two friends. Salina wants to buy 17 kg wheat, 3 kg pulses and 250gm ghee; while Rakhi wants to buy 15 kg wheat, 2kg pulses and 500 gm ghee. The prices of wheat, pulses and ghee per kg respectively are Rs. 8.00, Rs. 27.00 and Rs. 90.00. How much money will each spend? Clearly, the money needed by Salina and Rakhi will be :

Salina

Cost of 17 kg wheat $\Rightarrow 17 \times$ Rs. 8 = Rs. 136.00

Cost of 3 kg pulses $\Rightarrow 3 \times$ Rs. 27 = Rs. 81.00

Cost of 250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 90 = Rs. 22.50

Total = Rs. 239.50

Rakhi

Cost of 15 kg wheat $\Rightarrow 15 \times$ Rs. 8 = Rs. 120.00

Cost of 2 kg pulses $\Rightarrow 2 \times$ Rs. 27 = Rs. 54.00

Cost of 500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 90 = Rs. 45.00

Total = Rs. 219.00

In matrix form, the above information can be represented as follows:

Requirements Price Money Needed

$$\begin{bmatrix} \text{Wheat} & \text{pulses} & \text{ghee} \\ 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8 \\ 27 \\ 90 \end{bmatrix} = \begin{bmatrix} 17 \times 8 + 3 \times 27 + 0.250 \times 90 \\ 15 \times 8 + 2 \times 27 + 0.500 \times 90 \end{bmatrix} = \begin{bmatrix} 239.50 \\ 219.00 \end{bmatrix}$$

Another shop in the same locality quotes the following prices.

Wheat : Rs. 9 per kg.; pulses : Rs.26 per kg; ghee : Rs. 100 per kg.

The money needed by Salina and Rakhi to buy the required quantity of articles from this shop will be

Salina

17 kg wheat $\Rightarrow 17 \times$ Rs. 9 = Rs. 153.00

3 kg pulses $\Rightarrow 3 \times$ Rs. 26 = Rs. 78.00

250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 100 = Rs. 25.00

Total = Rs. 256.00

Rakhi

15 kg wheat $\Rightarrow 15 \times$ Rs. 9 = Rs. 135.00

2 kg pulses $\Rightarrow 2 \times$ Rs. 26 = Rs. 52.00

500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 100= Rs. 50.00

Total = Rs. 237.00

In matrix form, the above information can be written as follows :

Requirements Price Money needed

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 9.00 \\ 26.00 \\ 100.00 \end{bmatrix} = \begin{bmatrix} 17 \times 9.00 + 3 \times 26.00 + 0.250 \times 100 \\ 15 \times 900 + 2 \times 26.00 + 0.500 \times 100 \end{bmatrix} = \begin{bmatrix} 256.00 \\ 237.00 \end{bmatrix}$$

To have a comparative study, the two information can be combined in the following way:

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8.00 & 9.00 \\ 27.00 & 26.00 \\ 90.00 & 100.00 \end{bmatrix} = \begin{bmatrix} 239.50 & 256.00 \\ 219.00 & 237.00 \end{bmatrix}$$

Let us see how and when we write this product :

i) The three elements of first row of the first matrix are multiplied respectively by the corresponding elements of the first column of the second matrix and added to give element of the first row and the first column of the product matrix. In the same way, the product of the elements of the second row of the first matrix to the corresponding elements of the first column of the second matrix on being added gives the element of the second row and the first column of the product matrix; and so on

ii) The number of column of the first matrix is equal to the number of rows of the second matrix so that the first matrix is compatible for multiplication with the second matrix.

Thus, If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$, then

$$A \times B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \times \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{bmatrix}$$

Definition : If A and B are two matrices of order $m \times p$ and $p \times n$ respectively, then their product will be a matrix C of order $m \times n$; and if a_{ij}, b_{ij} and c_{ij} are the elements of the ith row and jth column of the matrices A, B and C respectively, then

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example 3.16 If $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, then find:

(a) AB (b) BA

Is $AB = BA$?

Solution : Order of A is 1×3

Order of B is 3×1

$\therefore$ Number of columns of A = Number of rows of B

$\therefore$ AB exists

Now, $AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

$$= \left[1\times(-2)+(-1)\times 0+2\times 2\right] = \left[-2+0+4\right] = \left[2\right]$$

Thus, $AB = \left[2\right]$, a matix of order 1×1

Again, number of columns of B = number of rows of A.

$\therefore$ BA exists

Now,

$$BA = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2\times 1 & (-2)\times(-1) & (-2)\times 2 \\ 0\times 1 & 0\times(-1) & 0\times 2 \\ 2\times 1 & 2\times(-1) & 2\times 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

Thus, $BA = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$, a matrix of order 3×3

From the above, we find that $AB \neq BA$

Example 3.17 If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then find AB.

Solution : Here, number of columns of A = number of rows of B

$\therefore$ AB exists

$$\text{Now } AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}_{2\times 2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 1\times 1+2\times 1 \\ 3\times 3+(-1)\times 1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Remarks : Can we find BA? The answer is no, because the number of columns of B is not equal to the number of rows of A.

Thus, in Example 3.17, we find that AB exists, but BA does not exist.

Example 3.18 Find AB and BA, if possible for the matrices A and B:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution : Here, Number of columns of A $\neq$ Number of rows of B

$\therefore$ AB does not exist.

Further, Number of columns of B $\neq$ Number of rows of A

$\therefore$ BA does not exist.

Example 3.19 If $A=\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, then find AB and BA. Also find if AB=BA.

Solution : Here, Number of columns of A = Number of rows of B

$\therefore$ AB exists.

Further, Number of columns of B = Number of rows of A

$\therefore$ BA also exists.

$$\text{Now, } AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+2\times2 & 1\times1+2\times2 \\ -1\times2+0\times2 & -1\times1+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 1+4 \\ -2+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}_{2\times2}$$

$$\text{and } BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times(-1) & 2\times2+1\times0 \\ 2\times1+2\times(-1) & 2\times2+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+0 \\ 2-2 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}_{2\times2}$$

Thus, $AB \neq BA$

Remarks : We observe that AB and BA are of the same order 2×2, but still $AB \neq BA$.

Example 3.20 If $A=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, find AB and BA. Is $AB = BA$?

Solution : Here, both A and B are of order 2×2. So, both AB and BA exist. Now

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2} \text{ and}$$

$$BA = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times 2}$$

Here, both AB and BA are of the same order and $AB = BA$.

Hence, if two matrcies A and B are multiplied, then the following five cases arise:

(i) Both AB and BA exist, but are of different orders

(ii) Only one of the products AB or BA exists.

(iii) Neither AB nor BA exist.

(iv) Both AB and BA exist and are of the same order, but $AB \neq BA$.

(v) Both AB and BA exist and are of the same order. Also, $AB = BA$.

Example 3.21 If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, verify that $A^2 - 2A - 3I = O$

Solution: Here,

$$A^2 = AA = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9+0 & 0+0 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\text{and} \quad 3I = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^2 - 2A - 3I = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \left\{ \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-9 & 0-0 \\ 0-0 & 9-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, verified.

Example 3.22 Solve the matrix equation :

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Solution : Here,

$$\text{L.H.S.} = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ x + y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x - 3y \\ x + y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow 2x - 3y = 1; x + y = 3$$

Solving these equations, we get

$x = 2$ and $y = 1$

Example 3.23 If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then find AB

Soution : Here,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\times(-1)+1\times 1 & 1\times 1+1\times(-1) \\ 1\times(-1)+1\times 1 & 1\times 1+1\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 1-1 \\ -1+1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Remarks : From Example 3.23, we find that the product of two non-zero matrices may be a zero matrix, i.e. $A \neq O$ and $B \neq O$ may imply $AB = O$

Hence, we conclude that the product of two non-zero matrices can be a zero matrix, whereas in numbers, the product of two non-zero numbers is always non-zero.

Example 3.24 If $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, find A^2.

Solution : A^2

$$= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Remarks : From Example 3.24, we find that the square of a non-zero matrix may be a zero matrix, i.e. $A \neq 0$ may imply $A^2 = O$.

Example 3.25 For $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, find

(a) $(AB)C$ (b) $A(BC)$

Is $(AB)C = A(BC)$?

Solution : (a) $(AB)C$

$$= \left\{\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}\right\}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & 0-4 \\ 12-5 & 0+10 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -4 \\ 7 & 10 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6+0 & 0-12 \\ -7+0 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

(b) $A(BC)$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\left\{\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -4+0 & 0+0 \\ 1+0 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-2 & 0-12 \\ -12+5 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

From (a) and (b), we find that $(AB)C = A(BC)$, i.e., matrix multiplication is associative.

CHECK YOUR PROGRESS 3.5

1. If $A = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, find AB and BA. Is $AB=BA$?

2. If $A = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$, find AB and BA. Is AB = BA?

3. If $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} x & y & z \end{bmatrix}$, find AB and BA, whichever exists.

4. If $A = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$, find BA. Does AB exists?

5. If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

   (a) Does AB exist? Why? (b) Does BA exist? Why?

6. If $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$, find AB and BA. Is AB=BA?

7. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & 4 \\ 5 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, find AB and BA . Is AB=BA?

8. If $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, find AB and BA. Is AB=BA?

9. Find the values of x and y if

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

10. For $A=\begin{bmatrix}2 & 0\\1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\3 & 4\end{bmatrix}$, verify that AB=O

11. For $A=\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$, verify that $A^2-5A+I=O$, where I is a unit matrix of order 2.

12. If $A=\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 2\\-1 & 1\end{bmatrix}$ and $C=\begin{bmatrix}4 & -3\\-2 & 3\end{bmatrix}$, find :

(a) A(BC) (b) (AB)C (c) (A+B)C

(d) AC+BC (e) A^2-B^2 (f) (A–B) (A+B)

13. If $A=\begin{bmatrix}2 & -1\\3 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & -2\\2 & 2\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}$, find : (a) AC (b) BC

Is AC = BC ? What do you conclude?

14. If $A=\begin{bmatrix}-1 & 0\\1 & -2\end{bmatrix}$, $B=\begin{bmatrix}1 & -1\\2 & 0\end{bmatrix}$ and $C=\begin{bmatrix}3 & 8\\7 & -1\end{bmatrix}$, find :

(a) B+C (b) A (B+C) (c) AB (d) AC (e) AB+AC

What do you observe?

15. For matices $A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}2 & -3\\-1 & 0\end{bmatrix}$, verify that $(AB)'=B'A'$

16. If $A=\begin{bmatrix}-1 & 2\\2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}3\\3\end{bmatrix}$, find X such that AX = B.

17. If $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$, show that

$A^2-(a+d)A=(bc-ad)I$

18 If $A=\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$, is it true that

(a) $(A+B)^2=A^2+B^2+2AB$? (b) $(A-B)^2=A^2+B^2-2AB$?

(c) $(A+B)(A-B)=A^2-B^2$?

LET US SUM UP

- A rectangular array of numbers, arranged in the form of rows and columns is called a *matrix*. Each number is called an *element* of the matrix.
- The *order* of a matrix having 'm' rows and 'n' columns is $m \times n$.
- If the number of rows is equal to the number of columns in a matrix, it is called a square matrix.
- A diagonal matrix is a square matrix in which all the elements, except those on the diagonal, are zeroes.
- A unit matrix of any order is a diagonal matrix of that order whose all the diagonal elements are 1.
- Zero matrix is a matrix whose all the elements are zeroes.
- Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.
- A transpose of a matrix is obtained by interchanging its rows and columns.
- Matrix A is said to be symmetric if $A' = A$ and skew symmetric if $A' = -A$.
- Scalar multiple of a matrix is obtained by multiplying each elements of the matrix by the scalar.
- The sum of two matrices (of the same order) is a matrix obtained by adding corresponding elements of the given matrices.
- Difference of two matrices A and B is nothing but the sum of matrix A and the negative of matrix B.
- Product of two matrices A of order $m \times n$ and B of order $n \times p$ is a matrix of order $m \times p$, whose elements can be obtained by multiplying the rows of A with the columns of B element wise and then taking their sum.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. How many elements are there in a matrix of order

(a) 2×1 (b) 3×2 (c) 3×3 (d) 3×4

2. Construct a matrix of order 3 × 2 whose elements a_{ij} are given by

(a) $a_{ij} = i - 2j$ (b) $a_{ij} = 3i - j$ (c) $a_{ij} = i + \frac{3}{2}j$

3. What is the order of the matrix?

(a) $A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 6 & 1 \end{bmatrix}$

4. Find the value of *x*, *y* and *z* if

(a) $\begin{bmatrix} x & y \\ z & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} x-2 & 3 \\ 0 & y+5 \end{bmatrix} = \begin{bmatrix} 1 & z \\ y+z & 2 \end{bmatrix}$ (d) $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}$, find :

(a) A+B (b) 2A (c) 2A−B

6. Find X, if

(a) $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} + X = \begin{bmatrix} 10 & -2 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. Find the values of *a* and *b* so that

$$\begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} a-b & 2 & -2 \\ 4 & a & b \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 2a+b & 5 \end{bmatrix}$$

8. For matrices A, B and C

$$A=\begin{bmatrix}1 & 3\\ 0 & 2\\ 5 & 7\end{bmatrix}, B=\begin{bmatrix}2 & 1\\ 1 & 4\\ 3 & 7\end{bmatrix} \text{ and } C=\begin{bmatrix}5 & 6\\ 7 & 1\\ 4 & 1\end{bmatrix}$$

verify that $A+(B+C)=(A+B)+C$

9. If $A=\begin{bmatrix}-1 & 1 & 2\\ 2 & 3 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\ 2 & 4\\ 6 & 5\end{bmatrix}$, find AB and BA. Is AB= BA?

10. If $A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -2\\ 0 & 1\end{bmatrix}$, find AB and BA. Is AB = BA?

11. If $A=\begin{bmatrix}1 & -3 & -4\\ -1 & 3 & 4\\ 1 & -3 & -4\end{bmatrix}$, find A^2.

12. Find A (B+C), if

$$A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}, B=\begin{bmatrix}3 & -1 & 0\\ 0 & 1 & 2\end{bmatrix} \text{ and } C=\begin{bmatrix}-2 & 0 & 3\\ 4 & 0 & -3\end{bmatrix}$$

13. If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}x & 1\\ y & -1\end{bmatrix}$ and $(A+B)^2=A^2+B^2$, find the values of x and y.

14. Show that $A=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}$ satisfies the matrix equation $A^2+4A-2I=O$.

ANSWERS

CHECK YOUR PROGRESS 3.1

1. $\begin{bmatrix} 56 & 65 & 71 \\ 29 & 37 & 57 \end{bmatrix}$; $\begin{bmatrix} 56 & 29 \\ 65 & 37 \\ 71 & 57 \end{bmatrix}$ 2. $\begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$; $\begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 6 & 3 \\ 4 & 3 & 5 \end{bmatrix}$

4. (a) 6 (b) 12 (c) 8 (d) 12 (e) ab (f) *mn*

5. (a) 1 × 8; 2 × 4; 4 × 2; 8 × 1 (b) 1 × 5; 5 × 1
(c) 1 × 12; 2 × 6; 3 × 4 ; 4 × 3; 6 × 2; 12 × 1
(d) 1 × 16; 2 × 8; 4 × 4; 8 × 2; 16 × 1

6. (a) 4 (b) 5 (c) 4 × 5 (d) 20
(e) $a_{14} = 0$; $a_{23} = 7$; $a_{34} = -3$; $a_{45} = 1$ and $a_{33} = 3$

7. (a) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 4 & 2 & \frac{4}{3} \\ 9 & \frac{9}{2} & 3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \\ \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{bmatrix}$

8. (a) $\begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 10 \\ 10 & 20 \\ 15 & 30 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

CHECK YOUR PROGRESS 3.2

1. (a) G (b) B (c) A, D, E and F (d) A, D and F
(e) D and F (f) F (g) C

2. (a) $a = 2$, $b = 10$, $c = 6$, $d = -2$
(b) $a = 2$, $b = 3$, $c = 2$, $d = 5$

(c) $a = \frac{3}{2}$, $b = -2$, $c = 2$, $d = -4$

3. No 4. No

CHECK YOUR PROGRESS 3.3

1. (a) $\begin{bmatrix} 28 & 8 \\ 8 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & -2 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{7}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-21}{2} & -3 \\ -3 & \frac{-9}{2} \end{bmatrix}$

2. (a) $\begin{bmatrix} 0 & -5 & 10 \\ 15 & 5 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -6 \\ -9 & -3 & -12 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$ (d) $\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{-3}{2} & \frac{-1}{2} & -2 \end{bmatrix}$ 3. $\begin{bmatrix} 7 & 0 \\ -28 & -14 \\ 0 & 7 \end{bmatrix}$

4. (a) $\begin{bmatrix} 15 & 0 & 5 \\ 20 & -10 & 0 \\ -5 & 0 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 0 & -4 \\ -16 & 8 & 0 \\ 4 & 0 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{5}{3} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-3}{2} & 0 & \frac{-1}{2} \\ -2 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-5}{2} \end{bmatrix}$.

5. (a) $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 10 & 8 \\ 9 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & -6 \\ -2 & -1 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

CHECK YOUR PROGRESS 3.4

1. (a) $\begin{bmatrix} 3 & -2 \\ 8 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -3 \\ 13 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 14 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & -5 \\ 19 & 10 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 3 & 6 \\ -5 & 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 & -6 \\ 5 & -3 & -5 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 9 \\ -9 & 2 & 10 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & -11 & -15 \\ 11 & -10 & -10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & -6 & 3 \\ 5 & 5 & 3 \\ 6 & 5 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 3 \\ 3 & -7 & 1 \\ 2 & 5 & -7 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -2 & -3 \\ -3 & 7 & -1 \\ -2 & -5 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -14 & 9 \\ 14 & 9 & 8 \\ 16 & 15 & 14 \end{bmatrix}$ 4. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \\ 4 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 18 \\ 6 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 15 & 3 \\ 21 & 27 \end{bmatrix}$ (c) $\begin{bmatrix} 17 & 21 \\ 27 & 31 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-17}{5} & \frac{-21}{5} \\ \frac{-27}{5} & \frac{-31}{5} \end{bmatrix}$

7. (a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ 0 & -4 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$

We observe that $(A+B)' = B' + A'$

8. (a) $\begin{bmatrix} 0 & 2 \\ -5 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 8 \\ 3 & -7 \end{bmatrix}$

(e) $\begin{bmatrix} -2 & 3 \\ -8 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -5 & 9 \\ -16 & -3 \end{bmatrix}$

CHECK YOUR PROGRESS 3.5

1. $AB = \begin{bmatrix} -6 \end{bmatrix}$; $BA = \begin{bmatrix} 0 & 0 & 0 \\ -4 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $AB \neq BA$

2. $AB=\begin{bmatrix}2 & 2\\ 1 & -6\end{bmatrix}$; $BA=\begin{bmatrix}-3 & 13 & -4\\ 1 & -1 & -2\\ 2 & -6 & 0\end{bmatrix}$ $AB \neq BA$

3. $AB=\begin{bmatrix}ax & ay & az\\ bx & by & bz\end{bmatrix}$; BA does not exist.

4. $BA=\begin{bmatrix}-2\\ -1\end{bmatrix}$; AB does not exist.

5. Both AB and BA do not exist. AB does not exist since the number of columns of A is not equal to the number of rows of B. BA also does not exist since number of coluumns of B is not equal to the number of rows of A.

6. $AB=\begin{bmatrix}0 & 5\\ 6 & 15\end{bmatrix}$; $BA=\begin{bmatrix}-2 & -1\\ 4 & 17\end{bmatrix}$; $AB \neq BA$

7. $AB=\begin{bmatrix}4 & -3 & 7\\ 3 & 17 & 24\\ 14 & -13 & 17\end{bmatrix}$; $BA=\begin{bmatrix}16 & -8 & -11\\ 16 & 11 & 3\\ 10 & 21 & 11\end{bmatrix}$; $AB \neq BA$.

8. $AB=\begin{bmatrix}10 & 0\\ 0 & -1\end{bmatrix}$; $BA=\begin{bmatrix}10 & 0\\ 0 & -1\end{bmatrix}$; $AB \neq BA$.

9. (a) $x=3,\ y=-1$ (b) $x=-1,\ y=2$

12. (a) $\begin{bmatrix}-14 & 18\\ 2 & 6\end{bmatrix}$ (b) $\begin{bmatrix}-14 & 18\\ 2 & 6\end{bmatrix}$ (c) $\begin{bmatrix}2 & 6\\ 0 & 3\end{bmatrix}$

(d) $\begin{bmatrix}2 & 6\\ 0 & 3\end{bmatrix}$ (e) $\begin{bmatrix}5 & 0\\ 7 & 8\end{bmatrix}$ (f) $\begin{bmatrix}-2 & -3\\ 9 & 15\end{bmatrix}$

13. (a) $\begin{bmatrix}1 & 2\\ 4 & 8\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ 4 & 8\end{bmatrix}$; AC= BC

Here, A≠B and C≠O, yet AC= BC
i.e. cancellation law does not hold good for matrices.

14. (a) $\begin{bmatrix}4 & 7\\ 9 & -1\end{bmatrix}$ (b) $\begin{bmatrix}-4 & -7\\ -14 & 9\end{bmatrix}$ (c) $\begin{bmatrix}-1 & 1\\ -3 & -1\end{bmatrix}$ (d) $\begin{bmatrix}-3 & -8\\ -11 & 10\end{bmatrix}$

(e) $\begin{bmatrix} -4 & -7 \\ -14 & 9 \end{bmatrix}$

We observe that A(B+C) = AB+AC

16. $x = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ 18. (a) No (b) No (c) No

TERMIAL EXERCISE

1. (a) 2

   (b) 6

   (c) 9

   (d) 12

2. (a) $\begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{bmatrix}$

   (c) $\begin{bmatrix} \frac{5}{2} & 4 \\ \frac{7}{2} & 5 \\ \frac{9}{2} & 6 \end{bmatrix}$

3. (a) 3×1

   (b) 1×3

   (c) 3×2

   (d) 2×3

4. (a) $x = 1, \; y = 2, \; z = 3$

   (b) $x = 5, y = 1, z = 5$

   (c) $x = 3, \; y = -3 \; z = 3$

   (d) $x = 2, y = 1, z = 5$

5. (a) $\begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -4 \\ 8 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -8 \\ 9 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 6 & -7 \\ 4 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 4 & -3 \\ -3 & 0 & -1 \end{bmatrix}$

7. $a = \frac{3}{2}$ $b = -\frac{3}{2}$

9. $AB = \begin{bmatrix} 13 & 11 \\ 38 & 43 \end{bmatrix}; BA = \begin{bmatrix} 5 & 10 & 17 \\ 6 & 14 & 24 \\ 4 & 21 & 37 \end{bmatrix}; AB \neq BA$

10. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; AB = BA$

11. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 9 & 1 & 1 \\ -1 & -4 & 10 \end{bmatrix}$

13. $x = 1, y = -4.$

4

DETERMINANTS AND THEIR APPLICATIONS

Every square matrix is associated with a unique number called the determinant of the matrix. There are numerous applications of determinants, the most famous of which is a method of solving system of 'n' linear equations in 'n' variables called Cramer's rule, named after **Gabriel Cramer** (1704-1752).

In this lesson, we will learn various properties of determinants and also evaluate determinants by different methods. We will also learn to solve system of linear equations by Cramer's rule using determinant.

OBJECTIVES

After studying this lesson, you will be able to :

- define determinant of a square matrix;
- define the minor and the cofactor of an element of a matrix;
- find the minor and the cofactor of an element of a matrix;
- find the value of a given determinant of order not exceeding 3;
- state the properties of determinants;
- evaluate a given determinant of order not exceeding 3 by using expansion method and using SARRUS diagram;
- state the conditions for a system of linear equations in two/three variables to have a unique solution; and
- state and apply Cramer's rule to solve a system of linear equations.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of solution of equations
- Knowledge of number system including complex number
- Four fundamental operations on numbers and expressions

4.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

On solving this system of equations for x and y, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \text{ provided } a_1b_2 - a_2b_1 \neq 0$$

The number $a_1b_2 - a_2b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

4.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

where a_{11} belongs to the 1st row and 1st column

a_{12} belongs to the 1st row and 2nd column

a_{21} belongs to the 2nd row and 1st column

a_{22} belongs to the 2nd row and 2nd column

write the elements in the following manner :

$$\begin{tikzcd} a_{11} \arrow[dr] & a_{12} \\ a_{21} \arrow[ur] & a_{22} \end{tikzcd}$$

Multiply the elements by the arrow. The sign of the arrow going **downwards** is positive, i.e., $a_{11}\, a_{22}$ and the sign of the arrow going **upwards** is negative, i.e., $-a_{21}\, a_{12}$

Add these two products, i.e., $a_{11}\, a_{22} + (-a_{21}.a_{12})$ or $a_{11}\, a_{22} - a_{21}\, a_{12}$ which is the required value of the determinant.

Example 4.1 Evaluate :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution :

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)(a+b) - (2a)(2b)$

$= a^2+2ab+b^2 - 4ab$

$= a^2+b^2 - 2ab$

$= (a-b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2+x+1)(x-1) - (x^2-x+1)(x+1)$

$= (x^3-1) - (x^3+1)$

$= -2$

Example 4.2 Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution :

(i) Now, $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2-9) - x^2 - x = -x - 9$

According to the question,

$-x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now, $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2+4x - 4x - 2 - 2x^2 - x - 2x - 1$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the equation

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x-1) + 1(x-1) = 0$

or, $(2x+1)(x-1) = 0$

or, $x = 1, -\frac{1}{2}$

4.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has $(3)^2 = 9$ elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 ,we write a determinant of order3 as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or $|A|$, $|B|$ etc.

$$\Delta = |a_{ij}|, \text{ where } i = 1, 2, 3 \text{ and } j = 1, 2, 3$$

4.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix".
With the 1 × 1 matrix $[a]$, we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3 × 3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we associate the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)\, a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 4.3 If $A = \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix}$, find $|A|$

Solution : $|A| = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9$

Example 4.4 If $A = \begin{bmatrix} a+b & a \\ b & a-b \end{bmatrix}$, find $|A|$

Solution : $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix} = (a+b)(a-b) - b \times a = a^2 - b^2 - ab$

Note : 1. The determinant of a unit matrix I is 1.
2. A square matrix whose determinant is zero, is called the singular matrix.

4.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements a_{11}, a_{12} and a_{13} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternately, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign. Therefore, a_{11} has been assigned positive sign.

Note : We can expand the determinant using any row or column. The value of the determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 4.5 Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$= 1 \text{ x } (20-2) - 2 \times (10-3) + 3 \times (4-12)$
$= 18-14-24$
$= -20$

Example 4.6 Expand the determinant, by using the second column

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\, 3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$= -2 \times (3-4) + 1 \times (1-6) - 3 \times (2-9)$
$= 2 - 5 + 21$
$= 18$

CHECK YOUR PROGRESS 4.1

1. Find |A|, if

(a) $A = \begin{bmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{bmatrix}$ (b) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $A = \begin{bmatrix} \sin\alpha + \cos\beta & \cos\beta + \cos\alpha \\ \cos\beta - \cos\alpha & \sin\alpha - \sin\beta \end{bmatrix}$ (d) $\begin{bmatrix} a+bi & c+di \\ c-di & a-bi \end{bmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{bmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$ (d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

4.6 MINORS AND COFACTORS

4.6.1 *Minor of a_{ij} in |A|*

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in |A| is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of |A| and is denoted by M_{ij}. For example, minor of 3 in the determinant $\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 4.7 Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1st row and 1st column. Thus to find the minor of a_{11}, we delete the 1st row and 1st column of |A|

The minor M_{11} of a_{11} is given by

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \; ; \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \; ; \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

4.6.2 *Cofactors of a_{ij} in |A|*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$. It is usually denoted by C_{ij}. Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

Example 4.8 Find the cofactors of the elements a_{11}, a_{12}, and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution :

The cofactor of any element a_{ij} is $(-1)^{i+j} M_{ij}$, then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22}\, a_{33} - a_{32}\, a_{23})$$

$$= (a_{22}\, a_{33} - a_{32}\, a_{23})$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}\, a_{33} - a_{31}\, a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$$

and $C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32}\, a_{13} - a_{12}\, a_{33})$

Example 4.9 Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution : The elements of the second row are $a_{21}=5; a_{22}=2; a_{23}=4.$

$$\text{Minor of } a_{21} \text{ (i.e., 5)} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$\text{Minor of } a_{22} \text{ (i.e., 2)} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$$

$$\text{and Minor of } a_{23} \text{ (i.e., 4)} = \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1} M_{21} = -(48) = -48$$

$$C_{22} = (-1)^{2+2} M_{22} = +(-13) = -13$$

$$\text{and} \quad C_{23} = (-1)^{2+3} M_{23} = -(-42) = 42$$

4.7 VALUE OF A DETERMINANT USING COFACTOR

With the cofactor notation, we can give expansion of a determinant in terms of the elements of any one row or column and their corresponding cofactors.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

where C_{ij} is the cofactor of the element a_{ij}

Example 4.10 Evaluate $\Delta = \begin{vmatrix} 3 & 1 & 7 \\ -6 & 2 & -3 \\ 8 & 4 & 5 \end{vmatrix}$ in terms of the elements of first row and their corresponding cofactors, and then in terms of the elements of first column and their corresponding cofactors. Verify that both the results are same.

Solution :

Expanding the determinant in terms of the elements of first row and their corresponding cofactors, we have the value of Δ, if we expand by R_1,

i.e. first row is given by

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

First we need to evaluate C_{11}, C_{12} and C_{13}

$$C_{11} = (-1)^{1+1}M_{11} \qquad = (-1)^2 \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$$

$$= 10+12 = 22$$

$$C_{12} = (-1)^{1+2}M_{12} \qquad = (-1) \begin{vmatrix} -6 & -3 \\ 8 & 5 \end{vmatrix}$$

$$= -(-30+24) = 6$$

$$C_{13} = (-1)^{1+3}M_{13} = (-1)^{1+3} \begin{vmatrix} -6 & 2 \\ 8 & 4 \end{vmatrix}$$

$$= -24 - 16 = -40$$

$$\therefore\ \Delta \quad = 3 \times 22 + 1 \times 6 + 7 \times (-40) = 66 + 6 - 280 = -208$$

Expanding the deteminant Δ in terms of the elements of first column and their corresponding cofactors, we have

$$\Delta = a_{11}\, C_{11} + a_{21}\, C_{21} + a_{31}\, C_{31}$$

$$\text{Now,} \quad C_{11} = (-1)^{1+1}\, M_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 10 + 12 = 22$$

$$C_{21} = (-1)^{2+1}\, M_{21} = (-1)^3 \begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix} = -(5-28) = 23$$

$$C_{31} = (-1)^{3+1}\, M_{31} = (-1)^4 \begin{vmatrix} 1 & 7 \\ 2 & -3 \end{vmatrix} = (-3-14) = -17$$

$$\therefore\ \Delta \quad = 3 \times 22 + (-6) \times 23 + 8 \times (-17) = 66-138-136 = -208$$

The value of the determinant by both the methods of expansion is -208. Thus, we verify that both the results are the same.

Example 4.11 Show that the sum of the products of the elements of one row (or column) by the cofactors of another row (or column) is always zero.

Solution :

Consider, $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

Here, $C_{11} = 5$ and $C_{12} = -4$

Let us find out $a_{21} \times C_{11} + a_{22} \times C_{12}$

$$= 4 \times 5 + 5 \times (-4)$$

$$= 20 - 20 = 0$$

Similarly, if we consider

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

Here $C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$

Let us find out

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

$$= 1 \times (-1) + 2(-7) + 3 \times (5)$$

$$= -1 - 14 + 15 = 0$$

We can verify this property for any other row or column.

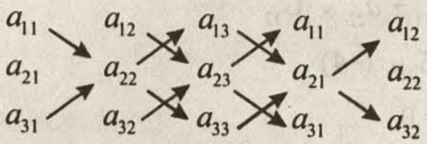
4.8 SARRUS DIAGRAM FOR THE EXPANSION OF DETERMINANT OF ORDER 3

Another simple method for evaluating a determinant of order 3 is Sarrus diagram.

Consider the determinant :

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

First of all we write down the three columns of the determinant, and then write the first row columns again as shown below


a_{11} a_{12} a_{13} a_{11} a_{12}
a_{21} a_{22} a_{23} a_{21} a_{22}
a_{31} a_{32} a_{33} a_{31} a_{32}

From the sum of terms connected with downward arrows, we subtract the terms connected with upwards arrows. Thus we get

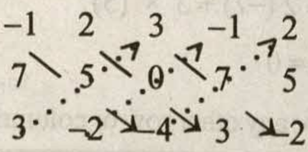
$$\begin{aligned} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \\ = \quad & a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\ = \quad & a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

which is equal to the value of given determinant.

Example 4.12 Using Sarrus diagram, find the value of the determinant

$$\Delta = \begin{vmatrix} -1 & 2 & 3 \\ 7 & 5 & 0 \\ 3 & -2 & -4 \end{vmatrix}$$

Solution :



The value of the determinant

$= (-1)(5)(-4) + 2.0.3 + 3.7(-2) - 3.5.3 - (-2).0.(-1) - (-4).7.2$

$= 20 + 0 - 42 - 45 - 0 + 56$

$= -11$

CHECK YOUR PROGRESS 4.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinant

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Using Sarrus diagram, evaluate each of the following determinants:

(a) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 5 \\ -1 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} -1 & -a & b \\ a & -1 & c \\ b & c & 1 \end{vmatrix}$

5. Using Sarrus diagram, show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

6. Solve for x, the following equations:

$$\text{(a)}\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0 \qquad \text{(b)}\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0 \qquad \text{(c)}\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$$

4.9 PROPERTIES OF DETERMINANTS

We shall now discuss some of the properties of determinants. These properties will help us in expanding the determinants.

Property 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first column, we have

$$\Delta \quad = 2\begin{vmatrix} -3 & 0 \\ 2 & -1 \end{vmatrix} - 0\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix}$$

$$= 2\,(3-0) - 0\,(1-6) + 4\,(0+9)$$

$$= 6 + 36 = 42$$

Let Δ' be the determinant obtained by interchanging rows and columns of Δ. Then

$$\Delta' = \begin{vmatrix} 2 & 0 & 4 \\ -1 & -3 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

Expanding the determinant Δ' by second column, we have (Recall that a determinant can be expanded by any of its rows or columns)

$$(-)\,0\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-)\,0\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= 0 + (-3)\,(-2-12) + 0$$

$$= 42$$

Thus, we see that $\Delta = \Delta'$

Property 2: If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= 2\,(4-3) - 3\,(2-9) + 1\,(1-6)$$

$$= 2 + 21 - 5 = 18$$

Let Δ' be the determinant obtained by interchanging C_1 and C_2

$$\text{Then } \Delta' = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

Expanding the determinant Δ' by first row, we have

$$\Delta' = 3\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3\,(2-9) - 2(4-3) + 1(6-1)$$

$$= -21 - 2 + 5 = -18$$

Thus we see that $\Delta' = -\Delta$

Corollary

If any row (or a column) of a determinant is passed over 'n' rows (or columns), then the resulting determinant Δ' is $\Delta = (-1)^n \Delta$

For example,

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 5 & 6 \\ 0 & 4 & 2 \end{vmatrix} = (-1)^2 \begin{vmatrix} 1 & 5 & 6 \\ 0 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 2\,(10-24) - 3\,(2-0) + 5(4)$$

$$= -28 - 6 + 20 = -14$$

Property 3: If any two rows (or columns) of a determinant are identical then the value of the determinant is zero.

Proof : Let $\Delta = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$

be a determinant with identical columns C_1 and C_2 and let Δ'determinant obtained from Δ by interchanging C_1 and C_2

Then,

$$\Delta' = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}$$

which is the same as Δ, but by property 2, the value of the determinant changes in sign, if its any two adjacent rows (or columns) are interchnaged

Therefore $\quad \Delta' = -\Delta$

Thus, we find that $\Delta = -\Delta$

or $\quad 2\Delta = 0 \Rightarrow \Delta = 0$

Hence the value of a determinant is zero, if it has two identical rows (or columns).

Property 4: If each element of a row (or column) of a determinant is multiplied by the same constant say, $k \neq 0$, then the value of the determinat is multiplied by that constant k.

$$\text{Let } \Delta = \begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$$

Expanding the determinant by first row, we have

$$\Delta = 2(3-0) - 1\,(0-0) + (-5)\,(0+12\,)$$
$$= 6 - 60 = -54$$

Let us multiply column 3 of Δ by 4. Then, the new determinant $\Delta'$is :

$$\Delta' = \begin{vmatrix} 2 & 1 & -20 \\ 0 & -3 & 0 \\ 4 & 2 & -4 \end{vmatrix}$$

Expanding the determinant $\Delta'$by first row, we have

$$\Delta' = 2\,(12-0) -1\,(0-0) + (-20)\,(0+12)$$
$$= 24 - 240 = -216 = 4(-54)$$
$$= 4\,\Delta$$

Corollary :

If any two rows (or columns) of a determinant are proportional, then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & ka_1 \\ a_2 & b_2 & ka_2 \\ a_3 & b_3 & ka_3 \end{vmatrix}$

Note that elements of column 3 are *k* times the corresponding elements of column 1

By Property 4, $\Delta = k \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$

$= k \times 0$ (by Property 3)

$= 0$

Property 5: If each element of a row (or of a column) of a determinant is expressed as the sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order whose remaining rows (or columns) do not change.

Proof : Let $\Delta = \begin{vmatrix} a_1 + \alpha & b_1 + \beta & c_1 + \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then, on expanding the determinant by the first row, we have

$$\Delta = (a_1 + \alpha)(b_2c_3 - b_3c_2) - (b_1 + \beta)(a_2c_3 - a_3c_2) + (c_1 + \gamma)(a_2b_3 - a_3b_2)$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) + \alpha(b_2c_3 - b_3c_2)$$

$$-\beta(a_2c_3 - a_3c_2) + \gamma(a_2b_3 - a_3b_2)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Thus, the determinant Δ can be expressed as the sum of the determinants of the same order.

Property 6: The value of a determinant does not change, if to each element of a row (or a column) be added (or subtracted) the some multiples of the corresponding elements of one or more other rows (or columns)

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Δ' be the determinant obtained from Δ by corresponding elements of R_3

i.e. $R_1 \to R_1 + kR_3$

$$\text{Then, } \Delta' = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

or, $\Delta' = \Delta + k \times 0$ (Row 1 and Row 3 are identical)

$\Delta' = \Delta$

4.10 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a determinant easily by applying the aforesaid properties. The purpose of simplification of a determinant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the determinant by that row (or column). We denote 1^{st}, 2^{nd} and 3^{rd} row by R_1, R_2, and R_3 respectively and 1st,2nd and 3rd column by C_1, C_2 and C_3 respectively.

Example 4.13 Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

Solution : $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \rightarrow C_1 + (C_2 + C_3)$

$$\therefore \quad \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \quad \text{(on expanding by } C_1\text{)}$$

(since w is a non-real cube root of unity, therefore, $1+w+w^2=0$)

Example 4.14 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)$

Solution : $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} \quad [R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3]$$

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & a(c-b) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c)\begin{vmatrix} 0 & 1 & -b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1, we have

$$\Delta \quad = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix}$$

$$= (a-c)(b-c)(b-a)$$

$$= (a-b)(b-c)(c-a)$$

Example 4.15

Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4\,abc$

Solution : $\Delta \quad = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad [R_1 \rightarrow R_1 - (R_2 + R_3)]$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\,[b(a+b) - bc] - 2b[bc - c(c+a)]$$

$$= 2bc\,[a+b-c] - 2bc[b-c-a]$$

$$= 2bc\,[(a+b-c) - (b-c-a)]$$

$$= 2bc\,[a+b-c-b+c+a]$$

$$= 4abc$$

Example 4.16 Evaluate:

$$\Delta \quad = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$= 0$, on expanding by C_1

Example 4.17 Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution : $$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \quad [C_3 \to C_2 + C_3]$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab+bc+ca) \times 0$ (by Property 3)

$= 0$

Example 4.18 Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : $$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= \text{abc}\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \begin{bmatrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{bmatrix}$$

$$= abc(-a)\begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)(-4bc)$$

$$= 4a^2b^2c^2$$

Example 4.19 Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2\,(a+3)$$

Solution : $$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (a+3)\begin{vmatrix}1 & 1 & 1\\ 1 & 1+a & 1\\ 1 & 1 & 1+a\end{vmatrix}$$

$$= (a+3)\begin{vmatrix}1 & 0 & 0\\ 1 & a & 0\\ 1 & 0 & a\end{vmatrix} \quad \begin{bmatrix}C_2 \to C_2 - C_1\\ C_3 \to C_3 - C_1\end{bmatrix}$$

$$= (a+3) \times (1)\begin{vmatrix}a & 0\\ 0 & a\end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3)$$

CHECK YOUR PROGRESS 4.3

1. Show that $\begin{vmatrix}x+3 & x & x\\ x & x+3 & x\\ x & x & x+3\end{vmatrix} = 27\,(x+1)$

2. Show that $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix} = bc+ca+ab+abc$

4. Show that $\begin{vmatrix}a & a+b & a+2b\\ a+2b & a & a+b\\ a+b & a+2b & a\end{vmatrix} = 9b^2\,(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for x :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

4.11 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY CRAMER'S RULE

Consider the system of linear equations

$$a_1x+b_1y = c_1$$
$$a_2x+b_2y = c_2$$

In this section, we shall learn to solve simultaneous linear equations in two unknowns and three unknowns with the help of determinants.

A pair of values of x and y which satisfies both the equations simultaneously is called a solution of the given system of linear equations and then the system is said to be consistent.

When the constants c_1 and c_2 on the R.H.S. of the equations are both zero, the system is said to be homogeneous system: otherwise, the system is said to be non-homogeneous. When the system is homogeneous, it has always a solution x = y = 0. It is called a trival solution.

If the determinant $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then the system of homogeneous equations has only one solution, i.e., the trival solution x = y = 0. However, if the determinant $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$, then the system forms a pair of dependent equations and in that case the system has infinite number of solutions, i.e., the system has also non-trival solutions. Determinants are used mainly to solve non-homogeneous linear equations in two or more variables.In this section, we shall learn to solve simultaneous linear equations in two/three unknowns with the help of determinants.The method of solving simultaneous linear equations by determinants is commonly known as Cramer's Rule.

4.11.1 Solving a system of Linear Equation in two variables

Consider the following system of linear equations :

$$2x + 3y - 5 = 0 \qquad \text{......(i)}$$

$$3x + 5y - 7 = 0 \qquad \text{......(ii)}$$

Usually, to find solutions of such a system, we apply the method of elimination. Thus, first of all solving for y, we get

$$3(2x + 3y - 5) = 0$$

$$2(3x + 5y - 7) = 0$$

$$3(2x) + 3(3y) = 3(5) \qquad \text{......}(iii)$$

$$2(3x) + 2(5y) = 2(7) \qquad \text{.....}(iv)$$

Subtracting (iii) from (iv), we get

$$\{2(5) - 3(3)\}\, y = 2(7) - 3(5)$$

i.e., $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} y = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$

$$\Rightarrow y = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \qquad \text{......(A)}$$

Observe that in the determinant of the numerator, the 1st column consists of the coefficeints of x and the 2nd columns consist of the constant terms.

And, in the determinant of the denominator of y, the 1st column and the 2nd column consist of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for x, we get

$$5(2x + 3y - 5) = 0$$

$$3(3x + 5y - 7) = 0$$

$$\Rightarrow \begin{cases} 5(2x) + 5(3y) - 5(5) = 0 & \text{......}(v) \\ 3(3x) + 3(5y) - 3(7) = 0 & \text{.....}(vi) \end{cases}$$

Subtracting (vi) from (v), we get

$$\{5(2) - 3(3)\}x = \{5(5) - 3(7)\}$$

i.e. $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} x = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}$

$$\Rightarrow x = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \qquad \text{......(B)}$$

Again observe that in the determinant of the numerator of x, the 1st column consists of the constant terms and the 2nd column consists of the coefficients of y.

And, in the determinant of the denominator of x, the 1st column and the 2nd column consist of the coefficients of x and y respectively.

$$\text{Thus,} \quad x = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{D_1}{D} \text{ (say)}$$

$$\text{and} \quad y = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \frac{D_2}{D} \text{ (say)}$$

are the solutions of the given system of equations in the determinant form.

Consider another system of equations:

$x + y = 3$ ...(i)

$2x - 3y = 1$ ...(ii)

Again, solving (i) and (ii) by elimination method, for x, we get

$$-3(x+y) = -3(3)$$

$$1(2x-3y) = 1$$

$$\Rightarrow \begin{cases} -3(x) - 3(y) = -3(3) & \text{...(iii)} \\ 1(2x) - 1(3y) = 1(1) & \text{...(iv)} \end{cases}$$

Subtracting (iv) from (iii), we get

$$-3(x) - 1(2x) = -3(3) - 1(1)$$

$$\Rightarrow \{-3(1) - 1(2)\}x = -3(3) - 1(1)$$

i.e. $$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} x = \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$$

$$\Rightarrow x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{D_1}{D} \text{ (say)}$$

Again, we observe that in the determinant D_1, the 1st column consists of the constant terms and the 2nd column consists of the coefficient of y. Also, in D, the 1st and the 2nd columns consist of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for y, we get

$$2(x) + 2(y) = 2(3) \qquad \text{...(v)}$$

$$1(2x) + 2(-3y) = 1(1) \qquad \text{...(vi)}$$

Subtracting (v) from (vi), we get

$$\{1(-3) - 2(1)\}y = 1(1) - 2(3)$$

i.e., $$\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} y = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$\Rightarrow y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{D_2}{D} \text{ (say)}$$

Again, we observe that the determinant D is nothing but the determinant of the coefficient of x and y and the determinant D_2 is obtained by replacing the coefficient of y by the constant terms.

Thus, $x = \dfrac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}$ and $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}$

is the solution of the given system of equations.

Therefore, if $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$

is a given system of linear equations in two variables, then its solution will be

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_1}{D}$$

and $$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_2}{D}$$

provided $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

4.11.2 Solution of a System of Linear Equations in Three Variables

Consider the following system of equations in three variables

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + y + z &= 1 \\ 3x + 3y + 5z &= 3 \end{aligned} \qquad \text{...(C)}$$

Let $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$

i.e., D is the determinant of the coefficients of x, y and z.

Then, $$xD = \begin{vmatrix} x & 2 & 3 \\ 2x & 1 & 1 \\ 3x & 3 & 5 \end{vmatrix}$$

Multiplying the 2nd column by y and the 3rd column by z and adding these to the 1st column, we get

$$xD = \begin{vmatrix} x+2y+3z & 2 & 3 \\ 2x+y+z & 1 & 1 \\ 3x+3y+5z & 3 & 5 \end{vmatrix}$$

From (C), we get

$$xD = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = D_1 \qquad \text{(say)}$$

i.e., $$xD = D_1 \Rightarrow x = \frac{D_1}{D}$$

Similarly, we will get

$$D_2 = \begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

Then, as before, we can see that

$$yD = D_2 \text{ and } zD = D_3$$

Thus, $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$, where $D \neq 0$

are the solutions of the given system of linear equations in three variables. Therefore, if

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is a given system of linear equations in three variables, then

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_1}{D}$$

$$y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_2}{D}$$

and $$z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_3}{D}$$

are the solutions of the given system, provided

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\neq 0$$

The method used for solving the system of equations in three variables can be used in exactly the same way to solve a system of 'n' equations in 'n' unknowns.
The method discussed above is commonly known as Cramer's rule, after the Swiss Mathematician Gabriel Cramer (1704-1752).

Note: Cramer's Rule does not apply if $D = 0$

Example 4.20 Solve the following system of equations by Cramer's Rule:

$$2x+3y=5$$
$$3x+5y=7$$

Solution:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$$

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix} = 25 - 21 = 4$$

and $$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} = 14 - 15 = -1$$

Thus, by Cramer's Rule $$x = \frac{D_1}{D} = \frac{4}{1} = 4$$

and $$y = \frac{D_2}{D} = \frac{-1}{1} = -1$$

Therefore, the solution of the given system of equations is $x = 4, y = -1$

Example 4.21 Solve the following system of equations by Cramer's Rule:

$$2x + y - 3z = 3$$
$$x + 2y + z = 5$$
$$3x - 5y + 2z = 1$$

Solution: We have.

$$D = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 2(4+5) - 1(2-3) - 3(-5-6) = 18 + 1 + 33 = 52 \neq 0$$

Also, to find D_1, the 1st column will be replaced by constants.

$$D_1 = \begin{vmatrix} 3 & 1 & -3 \\ 5 & 2 & 1 \\ 1 & -5 & 2 \end{vmatrix} = 3(4+5) - 1(10-1) - 3(-25-2) = 27 - 9 + 81 = 99$$

To find, D_2, 2nd column will be replaced by constants.

$$D_2 = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 5 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2(10-1) - 3(2-3) - 3(1-15) = 18 + 3 + 42 = 63$$

To find D_3, 3rd column will be replaced by constants

$$D_3 = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & -5 & 1 \end{vmatrix} = 2(2+25) - 1(1-15) + 3(-5-6) = 35$$

Thus, by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{99}{52}$$

$$y = \frac{D_2}{D} = \frac{63}{52}$$

and $$z = \frac{D_3}{D} = \frac{35}{52}$$

4.12 CONDITION FOR A SYSTEM OF LINEAR EQUATIONS TO HAVE A UNIQUE SOLUTION

Consider the system of equations

$$2x + 3y = 4$$
$$x - 2y = 3$$

Now $$D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4 - 3 = -7 \neq 0$$

Also $$D_1 = \begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix} = -8 - 9 = -17$$

and $$D_2 = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6 - 4 = 2$$

By Cramer's rule

$$x = \frac{D_1}{D} = \frac{-17}{-7} = \frac{17}{7}$$

and $$y = \frac{D_2}{D} = \frac{2}{-7} = \frac{-2}{7}$$

Thus, we find that for $D \neq 0$ and $D_1 \neq 0, D_2 \neq 0$, the given system of equations have non-zero, unique solution $x = \frac{17}{7}$ and $y = \frac{-2}{7}$

In this case, we say that the given system of equations is consistent.

Now, consider the system of equations

$$x + 2y = 0$$
$$-2x + 3y = 0$$

Here, $D = \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 3 + 4 = 7 \neq 0$

Also, $D_1 = \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0 - 0 = 0$

and $D_2 = \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix} = 0 - 0 = 0$

Hence, $x = \frac{D_1}{D} = \frac{0}{7} = 0$

and $\mathrm{y} = \frac{\mathrm{D}_2}{\mathrm{D}} = \frac{0}{7} = 0$

Thus, we find that for $D \neq 0$ and $D_1 = D_2 = 0$, the given system of equations will have only the trivial solution $x = y = 0$

We already know that Cramer's rule does not apply if $D = 0$. The two cases arise namely system of equations having (i) no solution, (ii) infinitely many solutions.

Consider the system of equations

$$2x + 4y = 5$$
$$x + 2y = 3$$

Here, $D = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4 - 4 = 0$

Since $D = 0$, Cramer's rule does not apply here.

Now, $D_1 = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 10 - 12 = -2$

and $D_2 = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1$

Hence, if D = 0 and $D_1 \neq 0, D_2 \neq 0$, the equations will have no solution.

Similar is the case for a system of three equations in three variables.For that, consider the system of equations

$$\begin{aligned} x+y+z&=2\\ x+2y+3z&=3\\ x+3y+5z&=5 \end{aligned} \qquad \text{...(i)}$$

Here, $D=\begin{vmatrix}1&1&1\\1&2&3\\1&3&5\end{vmatrix}=1(10-9)-1(5-3)+1(3-2)=1-2+1=0$

Now, $xD=\begin{vmatrix}x&1&1\\x&2&3\\x&3&5\end{vmatrix}$

Multiplying the 2nd column by *y* and the 3rd column by *z* and adding them to the 1st column, we get

$$xD=\begin{vmatrix}x+y+z&1&1\\x+2y+3z&2&3\\x+3y+5z&3&5\end{vmatrix}=\begin{vmatrix}2&1&1\\3&2&3\\5&3&5\end{vmatrix}=D_1 \qquad \text{[using(1)]}$$

Thus, $D_1=2(10-9)-1(15-15)+1(9-10)=2-0-1=1$

So, $xD=D_1 \Rightarrow xD=1$

$\Rightarrow \quad x=\dfrac{1}{D}=\dfrac{1}{0}$ which is undefined

Similarly, we get $yD=D_2$ and $zD=D_3$

where $D_2=\begin{vmatrix}1&2&1\\1&3&3\\1&5&5\end{vmatrix}=1(15-15)-2(5-3)+1(5-3)=0-4+2=-2$

and $D_3=\begin{vmatrix}1&1&2\\1&2&3\\1&3&5\end{vmatrix}=1(10-9)-1(5-3)+2(3-2)=1-2+2=1$

$$yD = D_2 \Rightarrow y = \frac{D_2}{D} \Rightarrow y = \frac{-2}{0} \text{ which is undefined.}$$

and $$zD = D_3 \Rightarrow z = \frac{D_3}{D} \Rightarrow z = \frac{1}{0} \text{ which is undefined.}$$

Thus, if $D = 0$ and $D_1 \neq 0, D_2 \neq 0$ and $D_3 \neq 0$, then the system of equations will have no solution. In this case, we say that the system of equations is inconsistent.

Now, consider the system of equations

$$\begin{aligned} x - y + 3z &= 6 \\ x + 3y - 3z &= -4 \\ 5x + 3y + 3z &= 10 \end{aligned}$$

Here, $D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 1(9+9) + 1(3+15) + 3(3-15) = 18 + 18 - 36 = 0$

Also $D_1 = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} = 6(9+9) + 1(-12+30) + 3(-12-30) = 108 + 18 - 126 = 0$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} = 1(-12+30) - 6(3+15) + 3(10+20) = 18 - 108 + 90 = 0$$

$$D_3 = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} = 1(30+12) + 1(10+20) + 6(3-15) = 42 + 30 = 72 = 0$$

Here,we say that for $D = 0$ and $D_1 = D_2 = D_3 = 0$ for the given system of equations has infinitely many solutions.

Consider the first two equations,

i.e., $$\begin{aligned} x - y + 3z &= 6 \\ x + 3y - 3z &= -4 \end{aligned}$$

These can be written as

$$x - y = 6 - 3z$$
$$x + 3y = -4 + 3z$$

Solving these equations by determinants, we get

$$x = \frac{\begin{vmatrix} 6-3z & -1 \\ -4+3z & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{3(6-3z)+1(-4+3z)}{3+1} = \frac{18-9z-4+3z}{4} = \frac{14-6z}{4}$$

$$x = \frac{7-3z}{2}$$

and $$y = \frac{\begin{vmatrix} 1 & 6-3z \\ 1 & -4+3z \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{1(-4+3z)-1(6-3z)}{3+1} = \frac{-4+3z-6+3z}{4} = \frac{-10+6z}{4}$$

$$y = \frac{-5+3z}{2}$$

Let $z = k$, where k is any number, then we get

$$x = \frac{7-3k}{2}, \quad y = \frac{-5+3k}{2} \text{ and } z = k$$

Thus, the system of equations has many solutions.

So, we conclude that for a given system of equations ($D_i = 1, 2, 3, ..., n$)

(i) If $D \neq 0$ and atleast one of $D_1, D_2, ..., D_n$ is not equal to zero, then the system will have non-zero, unique solution.

(ii) If $D \neq 0$ and each $D_i = 0$, then the system has only the trivial solution $x_1 = x_2 = ... = x_n = 0$

(iii) If $D = 0$ and some $D_i \neq 0$, then the system has no solution.

(iv) If $D = 0$ and each $D_i = 0$, then the system has infinitely many solutions.

Example 4.22 Solve the following system of equations:

$$x + y + z = 2$$
$$2x + 7y - 3z = 5$$
$$3x + 5y - z = 4$$

Solution : Here,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 7 & -3 \\ 3 & 5 & -1 \end{vmatrix} = 1(-7+15) - 1(-2+9) + 1(10-21) = 8 - 7 - 11 = -10 \neq 0$$

Now $D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 7 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 2(-7+15) - 1(-5+12) + 1(25-28) = 16 - 7 - 3 = 6$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 1(-5+12) - 2(-2+9) + 1(8-15) = 7 - 14 - 7 = -14$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 7 & 5 \\ 3 & 5 & 4 \end{vmatrix} = 1(28-25) - 1(8-15) + 2(10-21) = 3 + 7 - 22 = -12$$

Since $D \neq 0$ and $D_1 \neq 0,\ D_2 \neq 0,\ D_3 \neq 0$, therefore, the system of equations will have non-zero, unique solution.

Thus, by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{6}{-10} = \frac{-3}{5}$$

$$y = \frac{D_2}{D} = \frac{-14}{-10} = \frac{7}{5}$$

$$z = \frac{D_3}{D} = \frac{-12}{-10} = \frac{6}{5}$$

are the solutions of the given system of equations.

Example 4.23 Determine which of the following systems of equations will have a unique solution; and also find the solutions in each case:

(i) $2x-3y+4z=-9$
$-3x+4y+2z=-12$
$4x-2y-3z=-3$

(ii) $x+2y-z=0$
$2x+y+2z=0$
$x-3y+z=0$

(iii) $x+2y+z=2$
$2x+y+2z=3$
$x-3y+z=4$

(iv) $x+2y+3z=1$
$3x-y+2z=1$
$4x+y+5z=2$

Solution:

(i) $2x-3y+4z=-9$
$-3x+4y+2z=-12$
$4x-2y-3z=-3$

Here,

$$D=\begin{vmatrix}2 & -3 & 4\\ -3 & 4 & 2\\ 4 & -2 & -3\end{vmatrix}=2(-12+4)+3(9-8)+4(6-16)=-16+3-40=-53\neq 0$$

Also

$$D_1=\begin{vmatrix}-9 & -3 & 4\\ -12 & 4 & 2\\ -3 & -2 & -3\end{vmatrix}=-9(-12+4)+3(36+6)+4(24+12)=72+126+144=342$$

$$D_2=\begin{vmatrix}2 & -9 & 4\\ -3 & -12 & 2\\ 4 & -3 & 2\end{vmatrix}=2(36+6)+9(9-8)+4(9+48)=84+9+228=321$$

and

$$D_3=\begin{vmatrix}2 & -3 & -9\\ -3 & 4 & -12\\ 4 & -2 & -3\end{vmatrix}=2(-12-24)+3(9+48)-9(6-16)=-72+171+90=189$$

Since $D\neq 0$ and $D_1\neq 0$, $D_2\neq 0$, $D_3\neq 0$

$\therefore$ The system of equations will have a non-zero unique solution which is

$$x = \frac{D_1}{D} = \frac{342}{-53} = \frac{-342}{53}$$

$$y = \frac{D_2}{D} = \frac{321}{-53} = \frac{-321}{53}$$

and $$z = \frac{D_3}{D} = \frac{189}{-53} = \frac{-189}{53}$$

(ii) $x + 2y - z = 0$

$2x + y + 2z = 0$

$x - 3y + z = 0$

Here,

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 1(1+6) - 2(2-2) - 1(-6-1) = 7 + 7 = 14 \neq 0$$

Also,

$$D_1 = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -3 & 1 \end{vmatrix} = 0 \qquad \text{(expanding by the 1st column)}$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

and

$$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = 0$$

Thus, we find that $D \neq 0$ and $D_1 = D_2 = D_3 = 0$

$\therefore$ The system of linear equations will not have a unique solution. In fact, it has a trivial solution $x = 0, y = 0, z = 0$.

(iii)

$$x+2y+z=2$$
$$2x+y+2z=3$$
$$x-3y+z=4$$

Here $D=\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 2\\ 1 & -3 & 1\end{vmatrix}=0$

Also $D_1=\begin{vmatrix}2 & 2 & 1\\ 3 & 1 & 2\\ 4 & -3 & 1\end{vmatrix}=2(1+6)-2(3-8)+1(-9-4)=14+10-13=11$

$$D_2=\begin{vmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ 1 & 4 & 1\end{vmatrix}=0\ (\because C_1=C_3)$$

and $D_3=\begin{vmatrix}1 & 2 & 2\\ 2 & 1 & 3\\ 1 & -3 & 4\end{vmatrix}=1(4+9)-2(8-3)+2(-6-1)=13-10-14=-11$

Since $D=0$ and $D_1\neq 0,\ D_2=0$ and $D_3\neq 0$

$\therefore$ The system of equations has no solution.

(iv)

$$x+2y+3z=1$$
$$3x-y+2z=1$$
$$4x+y+5z=2$$

Here, $D=\begin{vmatrix}1 & 2 & 3\\ 3 & -1 & 2\\ 4 & 1 & 5\end{vmatrix}=1(-5-2)-2(15-8)+3(3+4)=-7-14+21=0$

Also, $D_1=\begin{vmatrix}1 & 2 & 3\\ 1 & -1 & 2\\ 2 & 1 & 5\end{vmatrix}=1(-5-2)-2(5-4)+3(1+2)=-7-2+9=0$

$$D_2=\begin{vmatrix}1 & 1 & 3\\ 3 & 1 & 2\\ 4 & 2 & 5\end{vmatrix}=1(5-4)-1(15-8)+3(6-4)=1-7+6=0$$

and $$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = 1(-2-1) - 2(6-4) + 1(3+4) = -3-4+7 = 0$$

Therefore, the given system of equations will have infinitely many solutions.

CHECK YOUR PROGRESS 4.4

1. Solve the following systems of equations by Cramer's Rule

   (a) $2x - 4y = 3$, $3x + y = 5$ (b) $x + 2y = 1$, $2x + 5y = 3$

2. Obtain the solutions of the systems of equations using Cramer's Rule

   (a) $2x + y + 3z = 1$, $x + 4y + 6z = 9$, $4x + 3y + 9z = 5$

   (b) $2x - 3y + 2z = 1$, $x + 3y - z = -2$, $x - y + 3z = 3$

   (c) $3x - 4y + 5z = -6$, $x + y - 2z = -1$, $2x + 3y + z = 5$

3. Solve the following systems of equations:

   (a) $3x + 2y = 4$, $2x + y = 3$ (b) $6x - 3y = -1$, $2x + 2y = -3$

   (c) $2x + 3y + 4z = 8$, $3x + y - z = -2$, $4x - y - 5z = -9$

   (d) $x + 3y - z = 4$, $3x - 2y + 5z = -4$, $5x - y - 4z = -9$

4. Determine which of the following systems of equations will have a unique solution. Also, find the solution in such a case.

   (a) $x - 2y = 4$, $-3x + 5y = -7$

   (b) $2x - y + z = 0$, $x + y - 2z = 0$, $3x + 2y - z = 0$

LET US SUM UP

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.
- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$

- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a *singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.
- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- The solution of a system of linear equations

  $$a_1x + b_1y + c_1z = d_1$$
  $$a_2x + b_2y + c_2z = d_2$$
  $$a_3x + b_3y + c_3z = d_3$$

  is given by $x = \frac{D_1}{D}$,$y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$

  $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and}$$

  $$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \text{ provided } D \neq 0$$

- The system of linear equations

  $$a_1x + b_1y + c_1z = d_1$$
  $$a_2x + b_2y + c_2z = d_2$$
  $$a_3x + b_3y + c_3z = d_3$$

(a) is consistent and has unique solution, when $D \neq 0$

(b) is inconsistent and has no solution, when $D=0$, and D_1, D_2, D_3 are not all zeros.

(c) is consistent and has infinitely many solutions, when $D = D_1 = D_2 = D_3 = 0$.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the first column.

3. Using Sarrus diagram, evaluate

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$$

4. Solve for x, if

$$\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$$

5. Using properties of determinants, show that

(a) $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate:

(a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

, w being an imaginary cube-root of unity

7. Using Cramer's rule, solve the following system of linear equations:

(a) $x-2y=3$
$2x+3y=4$

(b) $2x-3y=3$
$3x+2y=11$

(c) $x+y=2$
$4x-3y=3$

8. Using Cramer's rule, solve the following systems of equations:

(a) $2x+y+3z=8$
$3x+2y+3=z$
$x+3z=1+2y$

(b) $x-2y=3+z$
$2x+z=y$
$3-x=2z-3y$

9. Determine which of the following systems of equations will have a unique solution:

(a) $2x-6y+1=0$
$x-3y+2=0$

(b) $2x-3y=5$
$x-2y=6$

(c) $2x+3y+z=1$
$4x-6y+z=3$
$6x-3y+2z=5$

(d) $3x+y+2z=-1$
$x+2y-z=2$
$2x-y+3z=1$

ANSWERS

CHECK YOUR PROGRESS 4.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$

2. (a) and (d)

3. (a) 18 (b) -54 (c) $adf+2bce-ae^2-fb^2-de^2$
(d) $x-1$

CHECK YOUR PROGRESS 4.2

1. $M_{21}=39;\ C_{21}=-39$ **2.** $M_{13}=-5;\ C_{13}=-5$
$M_{22}=3;\ C_{22}=3$ $M_{23}=-7;\ C_{23}=7$
$M_{23}=-11;\ C_{23}=11$ $M_{33}=1;\ C_{33}=1$

3. (a) 19 (b) 0 (c) -131
(d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) 4 (b) 20 (c) $a^2+b^2+c^2+1$

6. (a) $x=2$ (b) $x=2,3$ (c) $x=2,-\frac{17}{7}$

CHECK YOUR PROGRESS 4.3

7. (a) a^3 (b) $2abc(a+b+c)^3$ **8.** $x=\frac{2}{3},\frac{11}{3},\frac{11}{3}$

CHECK YOUR PROGRESS 4.4

1. (a) $x=\frac{23}{14},\ y=\frac{1}{14}$ (b) $x=-1,\ y=1$

2. (a) $x=-1,y=2,z=\frac{1}{3}$ (b) $x=-\frac{3}{4},\ y=0,\ z=\frac{5}{4}$
(c) $x=-1,\ y=2,\ z=1$

3. (a) $x=2,\ y=-1$ (b) $x=-\frac{11}{8},\ y=-\frac{8}{9}$
(c) $x=1,\ y=-2,\ z=3$ (d) $x=-1,\ y=\frac{24}{13},\ z=\frac{7}{13}$

4. (a) Yes; $x = -6,\ y = -5$ (b) Yes; $x = 0,\ y = 0,\ z = 0$

(c) $x = 0,\ y = 3,\ z = 2$

TERMINAL EXERCISE

1. $M_{11} = -2,\ M_{12} = -1,\ M_{13} = 1,\ M_{21} = -7,\ M_{22} = -5,\ M_{23} = -1,$
$M_{31} = -8,\ M_{32} = -7,\ M_{33} = -2$

$C_{11} = -2,\ C_{12} = 1,\ C_{13} = 1,\ C_{21} = 7,\ C_{22} = -5,\ C_{23} = 1,$
$C_{31} = -8,\ C_{32} = 7,\ C_{33} = -2$

2. 0 **3.** –31 **4.** $x = 0,\ x = 1$

6. (a) –8 (b) 0

7. (a) $x = \frac{17}{7},\ y = -\frac{2}{7}$ (b) $x = 3,\ y = 1$ (c) $x = \frac{9}{7},\ y = \frac{5}{7}$

8. (a) $x = 2,\ y = 3,\ z = 3$ (b) $x = -\frac{1}{2},\ y = -\frac{3}{2},\ z = -\frac{1}{2}$

9. (b) only

5

INVERSE OF A MATRIX AND ITS APPLICATION

LET US CONSIDER AN EXAMPLE:

Abhinav spends Rs. 120 in buying 2 pens and 5 note books whereas Shantanu spends Rs. 100 in buying 4 pens and 3 note books. We will try to find the cost of one pen and the cost of one note book using matrices.

Let the cost of 1 pen be Rs. x and the cost of 1 note book be Rs. y. Then the above information can be written in matrix form as:

$$\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$$

This can be written as $AX = B$

where $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$

Our aim is to find $X = \begin{bmatrix} x \\ y \end{bmatrix}$

In order to find X, we need to find a matrix A^{-1} so that $X = A^{-1}B$

This matrix A^{-1} is called the inverse of the matrix A.

In this lesson, we will try to find the existence of such matrices. We will also learn to solve a system of linear equations using matrix method.

OBJECTIVES

After studying this lesson, you will be able to :

- define a minor and a cofactor of an element of a matrix;
- find minor and cofactor of an element of a matrix;
- find the adjoint of a matrix;
- define and identify singular and non-singular matrices;
- find the inverse of a matrix, if it exists;
- represent system of linear equations in the matrix form $AX = B$; and
- solve a system of linear equations by matrix method.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a determinant.
- Determinant of a matrix.
- Matrix with its determinant of value 0.
- Transpose of a matrix.
- Minors and Cofactors of an element of a matrix.

5.1 DETERMINANT OF A SQUARE MATRIX

We have already learnt that with each square matrix, a determinant is associated. For any given matrix, say $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$

its determinant will be $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$. It is denoted by $|A|$.

Similarly, for the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{bmatrix}$, the corresponding determinant is

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{vmatrix}$$

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is non-zero, i.e. $|A| \neq 0$

Example 5.1 Determine whether matrix A is singular or non-singular where

(a) $A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

Solution: (a) Here, $|A| = \begin{vmatrix} -6 & -3 \\ 4 & 2 \end{vmatrix}$

$$= (-6)(2) - (4)(-3)$$
$$= -12 + 12 = 0$$

Therefore, the given matrix A is a singular matrix.

(b)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix}$$

Here,

$$= 1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -7 + 4 - 3$$

$$= -6 \neq 0$$

Therefore, the given matrix is non-singular.

Example 5.2 Find the value of x for which the following matrix is singular:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$$

Solution: Here,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix}$$

$$= 1(-6-2) + 2(-3-x) + 3(2-2x)$$

$$= -8 - 6 - 2x + 6 - 6x$$

$$= -8 - 8x$$

Since the matrix A is singular, we have $|A| = 0$

$$|A| = -8 - 8x = 0$$

$$or \quad x = -1$$

Thus, the required value of x is -1.

Example 5.3 Given $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$. Show that $|A| = |A'|$, where A' denotes the transpose of the matrix.

Solution: Here, $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$

This gives $A' = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \quad \ldots(1)$

and $\quad |A'| = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \quad \ldots(2)$

From (1) and (2), we find that $|A| = |A'|$

5.2 MINORS AND COFACTORS OF THE ELEMENTS OF SQUARE MATRIX

Consider a matrix $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The determinant of the matrix obtained by deleting the *i*th row and *j*th column of *A*, is called the minor of a_{ij} and is denoted by M_{ij}.

Cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, M_{23} = Minor of a_{23}

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and C_{23} = Cofactor of a_{23}

$$= (-1)^{2+3} M_{23}$$

$$= (-1)^{5} M_{23}$$

$$= -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 5.4 Find the minors and the cofactors of the elements of matrix $A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$

Solution: For matrix A, $|A| = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} = 6 - 30 = -24$

M_{11} (minor of 2) = 3; $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = 3$

M_{12} (minor of 5) = 6; $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -6$

M_{21} (minor of 6) = 5; $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -5$

M_{22} (minor of 3) = 2; $C_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22} = 2$

Example 5.5 Find the minors and the cofactors of the elements of matrix

$$A = \begin{bmatrix} -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 1 & 3 \end{bmatrix}$$

Solution: Here, $M_{11} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15+2 = 17;\ C_{11} = (-1)^{1+1} M_{11} = 17$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6+8 = 14;\ C_{12} = (-1)^{1+2} M_{12} = -14$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2-20 = -18;\ C_{13} = (-1)^{1+3} M_{13} = -18$$

$$M_{21} = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9-6 = 3;\ C_{21} = (-1)^{2+1} M_{21} = -3$$

$$M_{22} = \begin{vmatrix} -1 & 6 \\ 4 & 3 \end{vmatrix} = (-3-24) = -27;\ C_{22} = (-1)^{2+2} M_{22} = -27$$

$$M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = (-1-12) = -13;\ C_{23} = (-1)^{2+3} M_{23} = 13$$

$$M_{31} = \begin{vmatrix} 3 & 6 \\ 5 & -2 \end{vmatrix} = (-6-30) = -36;\ C_{31} = (-1)^{3+1} M_{31} = -36$$

$$M_{32} = \begin{vmatrix} -1 & 6 \\ 2 & -2 \end{vmatrix} = (2-12) = -10;\ C_{32} = (-1)^{3+2} M_{32} = 10$$

and $M_{33} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-5-6) = -11;\ C_{33} = (-1)^{3+3} M_{33} = -11$

CHECK YOUR PROGRESS 5.1

1. Find the value of the determinant of following matrices:

(a) $A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$, (b) $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

2. Determine whether the following matrix are singular or non-singular.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & -6 \end{bmatrix}$ (b) $Q = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \end{bmatrix}$

3. Find the minors of the following matrices:

(a) $A = \begin{bmatrix} 3 & -1 \\ 7 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$

4. (a) Find the minors of the elements of the 2nd row of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 1 \end{bmatrix}$$

(b) Find the minors of the elements of the 3rd row of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & 0 & -3 \end{bmatrix}$$

5. Find the cofactors of the elements of each the following matrices:

(a) $A = \begin{bmatrix} 3 & -2 \\ 9 & 7 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 4 \\ -5 & 6 \end{bmatrix}$

6. (a) Find the cofactors of elements of the 2nd row of matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 4 & 1 & -2 \end{bmatrix}$$

(b) Find the cofactors of the elements of the 1st row of matrix

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -2 \\ -5 & -3 & 0 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 7 & 4 \end{bmatrix}$, verify that

(a) $|A| = |A'|$ and $|B| = |B'|$ (b) $|AB| = |A|\,|B| = |BA|$.

5.3 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ be a matrix. Then $|A| = \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix}$

Let M_{ij} and C_{ij} be the minor and cofactor of a_{ij} respectively. Then

$$M_{11} = |7| = 7; \; C_{11} = (-1)^{1+1}\,|7| = 7$$

$$M_{12} = |5| = 5; \; C_{12} = (-1)^{1+2}\,|5| = -5$$

$$M_{21} = |1| = 1; \; C_{21} = (-1)^{2+1}\,|1| = -1$$

$$M_{22} = |2| = 2; \; C_{22} = (-1)^{2+2}\,|2| = 2$$

We replace each element of A by its cofactor and get

$$B = \begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix} \quad \ldots(1)$$

The transpose of the matrix B of cofactors obtained in (1) above is

$$B' = \begin{bmatrix} 7 & -1 \\ -5 & 2 \end{bmatrix} \quad \ldots(2)$$

The matrix B' obtained above is called the adjoint of matrix A. It is denoted by Adj A.

Thus, adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix.

Working Rule: To find the Adj A of a matrix A:

(a) replace each element of A by its cofactor and obtain the matrix of cofactors; and

(b) take the transpose of the marix of cofactors, obtained in (a).

Example 5.6 Find the adjoint of

$$A = \begin{bmatrix} -4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution: Here, $|A| = \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$ Let A_{ij} be the cofactor of the element a_{ij}.

Then, $A_{11} = (-1)^{1+1}\ (-3) = -3 \qquad A_{21} = (-1)^{2+1}\ (5) = -5$

$A_{12} = (-1)^{1+2}\ (2) = -2 \qquad A_{22} = (-1)^{2+2}\ (-4) = -4$

We replace each element of A by its cofactor to obtain its matrix of cofators as

$$\begin{bmatrix} -3 & -2 \\ -5 & -4 \end{bmatrix} \quad \ldots(1)$$

Transpose of matrix in (1) is Adj A.

Thus, Adj $A = \begin{bmatrix} -3 & -5 \\ -2 & -4 \end{bmatrix}$

Example 5.7 Find the adjoint of $A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{bmatrix}$

Solution: Here,

$$\text{A} = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{vmatrix}$$

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$

Then $A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = (-4-2) = -6$; $A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = -(3-5) = 2$

$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} = (-6-20) = -26$; $\text{A}_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = -(1-4) = 3$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}1 & 2\\ 5 & -1\end{vmatrix} = (-1-10) = -11;\ A_{23} = (-1)^{2+3}\begin{vmatrix}1 & -1\\ 5 & 2\end{vmatrix} = -(2+5) = -7$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}-1 & 2\\ 4 & 1\end{vmatrix} = (-1-8) = -9;\ A_{32} = (-1)^{3+2}\begin{vmatrix}1 & 2\\ -3 & 1\end{vmatrix} = -(1+6) = -7$$

$$\text{and}\quad A_{33} = (-1)^{3+3}\begin{vmatrix}1 & -1\\ -3 & 4\end{vmatrix} = (4-3) = 1$$

Replacing the elements of A by their cofactors, we get the matrix of cofactors as

$$\begin{bmatrix}-6 & 2 & -26\\ 3 & -11 & -7\\ -9 & -7 & 1\end{bmatrix}\qquad \text{Thus, Adj } A = \begin{bmatrix}-6 & 3 & -9\\ 2 & -11 & -7\\ -26 & -7 & 1\end{bmatrix}$$

If A is any square matrix of order n, then $A(\text{Adj } A) = (\text{Adj } A)\, A = |A|\, I_n$ where I_n is the unit matrix of order n.

Verification:

(1) Consider $A = \begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$

Then $|A| = \begin{vmatrix}2 & 4\\ -1 & 3\end{vmatrix}$ or $|A| = 2 \times 3 - (-1) \times (4) = 10$

Here, $A_{11}=3$; $A_{12}=1$; $A_{21}=-4$ and $A_{22}=2$

Therefore, $\text{Adj A} = \begin{bmatrix}3 & -4\\ 1 & 2\end{bmatrix}$

Now, $$A(\text{Adj}A) = \begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}\begin{bmatrix}3 & -4\\ 1 & 2\end{bmatrix} = \begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix} = 10\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = |A|\, I_2$$

(2) Consider, $A = \begin{bmatrix}3 & 5 & 7\\ 2 & -3 & 1\\ 1 & 1 & 2\end{bmatrix}$

Then, $|A| = 3(-6-1) - 5(4-1) + 7(2+3) = -1$

Here, $A_{11} = -7$; $A_{12} = -3$; $A_{13} = 5$

$A_{21} = -3;\ A_{22} = -1;\ A_{23} = 2$

$A_{31} = 26;\ A_{32} = 11;\ A_{33} = -19$

Therefore, $$\text{Adj } A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$$

Now $$(A)(\text{Adj } A) = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

Also, $$(\text{Adj } A)\, A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$$

Note : If A is a singular matrix, i.e. $|A| = 0$, then A (Adj A) = O

CHECK YOUR PROGRESS 5.2

1. Find adjoint of the following matrices:

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

2. Find adjoint of the following matrices :

(a) $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$

Also verify in each case that $A(\text{Adj } A) = (\text{Adj } A)\, A = |A| I_2$.

3. Verify that

A(Adj A) = (Adj A) A = |A| I_3, where A is given by

(a) $\begin{bmatrix} 6 & 8 & -1 \\ 0 & 5 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 7 & 9 \\ 0 & -1 & 2 \\ 3 & -7 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

5.4 INVERSE OF A MATRIX

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will find, if possible, a matrix

$$B = \begin{bmatrix} x & y \\ u & v \end{bmatrix} \text{ such that } AB = BA = I$$

i.e., $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing both sides, we get

$$ax + bu = 1 \quad ay + bv = 0$$

$$cx + du = 0 \qquad cy + dv = 1$$

Solving for x, y, u and v, we get

$$x = \frac{d}{ad - bc},\ y = \frac{-b}{ad - bc},\ u = \frac{-c}{ad - bc},\ v = \frac{a}{ad - bc}$$

provided $ad - bc \neq 0$, i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$

Thus, $$B = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

or $$B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It may be verified that $BA = I$.

It may be noted from above that, we have been able to find a matrix.

$$B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|} Adj\ A \qquad ...(1)$$

This matrix B, is called the inverse of A and is denoted by A^{-1}.

For a given matrix A, if there exists a matrix B such that AB = BA = I, then B is called the multiplicative inverse of A. We write this as B= A^{-1}.

Note: Observe that if $ad - bc = 0$, i.e., $|A| = 0$, the R.H.S. of (1) does not exist and $B\ (=A^{-1})$ is not defined. This is the reason why we need the matrix A to be non-singular in order that A possesses multiplicative inverse. Hence only non-singular matrices possess multiplicative inverse. Also B is non-singular and $A = B^{-1}$.

Example 5.8 Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution : $$A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$$

Therefore, $|A| = -12 - 10 = -22 \neq 0$

$\therefore A$ is non-singular. It means A has an inverse. i.e. A^{-1} exists.

Now, $$\text{Adj}\ A = \begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}\ A = \frac{1}{-22}\begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{22} & \dfrac{5}{22} \\ \dfrac{1}{11} & -\dfrac{2}{11} \end{bmatrix}$$

Note : Verify that $AA^{-1} = A^{-1}A = I$

Example 5.9 Find the inverse of matrix

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$$

Solution : Here, $A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$

$$\begin{aligned} \therefore \quad |A| &= 3(5-24) - 2(-5-30) - 2(4+5) \\ &= 3(-19) - 2(-35) - 2(9) \\ &= -57 + 70 - 18 \\ &= -5 \neq 0 \end{aligned}$$

$\therefore$ A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} = 5 - 24 = -19,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 6 \\ 5 & -5 \end{vmatrix} = -(-5-30) = 35.$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix} = 4 + 5 = 9,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -2 \\ 4 & -5 \end{vmatrix} = -(-10+8) = 2$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & -2 \\ 5 & -5 \end{vmatrix} = -15 + 10 = -5,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = -(12 - 10) = -2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -2 \\ -1 & 6 \end{vmatrix} = 12 - 2 = 10,$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -2 \\ 1 & 6 \end{vmatrix} = -(18+2) = -20$$

and $$A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -3 - 2 = -5$$

Matrix of cofactors = $\begin{bmatrix} -19 & 35 & 9 \\ 2 & -5 & -2 \\ 10 & -20 & -5 \end{bmatrix}$. Hence AdjA= $\begin{bmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}.Adj\ A = \frac{1}{-5}\begin{bmatrix} -19 & 2 & 10 \\ 35 & -5 & -20 \\ 9 & -2 & -5 \end{bmatrix} = \begin{bmatrix} \frac{19}{5} & \frac{-2}{5} & -2 \\ -7 & 1 & 4 \\ \frac{-9}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

Note ; Verify that $A^{-1}A = AA^{-1} = I$,

Example 5.10 If A = $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and B= $\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$; find

(i) $(AB)^{-1}$ (ii) $B^{-1}A^{-1}$ (iii) Is $(AB)^{-1} = B^{-1}A^{-1}$?

Solution : (i) Here, AB = $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2+0 & 1+0 \\ -4+0 & 2+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

$$\therefore\quad |AB| = \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -6 + 4 = -2 \neq 0.$$

Thus, $(AB)^{-1}$ exists.

Let us denote AB by C

Let C_{ij} be the cofactor of the element c_{ij} of $|C|$.

Then, $C_{11} = (-1)^{1+1}(3) = 3$ $\quad C_{21} = (-1)^{2+1}(1) = -1$

$C_{12} = (-1)^{1+2}(-4) = 4$ $\quad C_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj}(C) = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

$$C^{-1} = \frac{1}{|C|}\text{Adj}(C) = \frac{1}{-2}\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(ii) To find $B^{-1}A^{-1}$, first we will find B^{-1}.

Now, $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ $\therefore |B| = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = 2 - 0 = 2 \neq 0$

$\therefore$ B^{-1} exists.

Let B_{ij} be the cofactor of the element b_{ij} of $|B|$

then $B_{11} = (-1)^{1+1}(-1) = -1$ $\quad (B_{21} = (-1)^{2+1}(1) = -1$

$B_{12} = (-1)^{1+2}(0) = 0$ and $\quad B_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj } B = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$

$$\therefore\ B^{-1} = \frac{1}{|B|} AdjB = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ Therefore, $|A| = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1 - 0 = -1 \neq 0$

Therefore, A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$

then $A_{11} = (-1)^{1+1}(-1) = -1$ $\quad A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(2) = -2$ and $\quad A_{22} = (-1)^{2+2}(1) = 1$

Hence, $\text{Adj A}=\begin{bmatrix}-1 & 0\\ -2 & 1\end{bmatrix}$

$$\Rightarrow \text{A}^{-1}=\frac{1}{|A|}\,Adj\ \text{A} = \frac{1}{-1}\begin{bmatrix}-1 & 0\\ -2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$$

Thus, $B^{-1}A^{-1}=\begin{bmatrix}\frac{-1}{2} & \frac{-1}{2}\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}$

$$=\begin{bmatrix}\frac{-1}{2}-1 & 0+\frac{1}{2}\\ 0-2 & 0+1\end{bmatrix}=\begin{bmatrix}\frac{-3}{2} & \frac{1}{2}\\ -2 & 1\end{bmatrix}$$

(iii) From (i) and (ii), we find that

$$(\text{AB})^{-1}=\text{B}^{-1}\text{A}^{-1}=\begin{bmatrix}\frac{-3}{2} & \frac{1}{2}\\ -2 & 1\end{bmatrix}$$

Hecne, $(\text{AB})^{-1}=\text{B}^{-1}\text{A}^{-1}$

CHECK YOUR PROGRESS 5.3

1. Find, if possible, the inverse of each of the following matrices:

(a) $\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 2\\ -3 & -4\end{bmatrix}$ (c) $\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}$

2. Find, if possible, the inverse of each of the following matrices :

(a) $\begin{bmatrix}1 & 0 & 2\\ 2 & 1 & 3\\ 4 & 1 & 2\end{bmatrix}$ (b) $\begin{bmatrix}3 & -1 & 2\\ 5 & 2 & 4\\ 1 & -3 & -2\end{bmatrix}$

Verify that $A^{-1}A=AA^{-1}=\text{I}$ for (a) and (b).

3. If $A=\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & 4\\ 3 & 1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 0\\ 1 & 4 & 3\\ 3 & 0 & -2\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$

4. Find $(A')^{-1}$ if $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$

5. If $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ and $B=\frac{1}{2}\begin{bmatrix}b+c & c-a & b-a\\ c-b & c+a & a-b\\ b-c & a-c & a+b\end{bmatrix}$

show that ABA^{-1} is a diagonal matrix.

6. If $\phi(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$, show that $[\phi(x)]^{-1}=\phi(-x)$

7. If $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$, show that $A'A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$

8. If $A=\begin{bmatrix}a & b\\ c & \dfrac{1+bc}{a}\end{bmatrix}$, show that $aA^{-1}=(a^2+bc+1)I-aA$

9. If $A=\begin{bmatrix}-1 & 2 & 0\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$, show that $A^{-1}=A^2$

10. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & -4 & 7\\ 1 & -8 & 4\end{bmatrix}$, show that $A^{-1}=A'$

5.5 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS.

In earlier classes, you have learnt how to solve linear equations in two or three unknowns (simultaneous equations). In solving such systems of equations, you used the process of elimination of variables. When the number of variables invovled is large, such elimination process becomes tedious.

You have already learnt an alternative method, called Cramer's Rule for solving such systems of linear equations.

We will now illustrate another method called the matrix method, which can be used to solve the system of equations in large number of unknowns. For simplicity the illustrations will be for system of equations in two or three unknowns.

5.5.1 MATRIX METHOD

In this method, we first express the given system of equation in the matrix form $AX = B$, where A is called the co-efficient matrix.

For example, if the given system of equation is $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, we express them in the matrix equation form as :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

If the given system of equations is $a_1x + b_1y + c_1z = d_1$ and

$a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$, then this system is expressed in the matrix equation form as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Before proceding to find the solution, we check whether the coefficient matrix A is non-singular or not.

or, $AX = B$ ...(i)

where $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

Now, $|A| = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = 2 \times 2 - 1 \times (-3)$

$$= 4 + 3 = 7 \neq 0$$

$\therefore$ A^{-1} exists

Since, $\text{Adj}\,(A) = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix} \quad \text{.... (ii)}$$

From (i), we have $X = A^{-1} B$

or, $X = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{23}{7} \\ \frac{-1}{7} \end{bmatrix}$

Thus, $x = \frac{23}{7}$, $y = \frac{-1}{7}$ is the solution of this system of equations.

Example 5.13 Solve the following system of equations, using matrix method.

$$\left.\begin{aligned} x + 2y + 3z &= 14 \\ x - 2y + z &= 0 \\ 2x + 3y - z &= 5 \end{aligned}\right\}$$

Solution : The given equations expressed in the matrix equation form as :

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix} \qquad \text{... (i)}$$

which is in the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix}$$

$\therefore \quad X = A^{-1} B$ ... (ii)

Here, $|A| = 1\,(2-3) - 2\,(-1-2) + 3\,(3+4)$

$= 26 \neq 0$

$\therefore \quad A^{-1}$ exists.

Also, Adj $A = \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

Hence, from (ii), we have $X = A^{-1}B = \frac{1}{|A|} AdjA.\,B$

$$X = \frac{1}{26}\begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}\begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix} 26 \\ 52 \\ 78 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Thus, $x=1, y=2$ and $z=3$ is the solution of the given system of equations.

Example 5.14 Solve the following system of equations, using matrix method :

$$x+2y+z=2$$
$$2x-y+3z=3$$
$$x+3y-z=0$$

Solution: The given system of equation can be represented in the matrix equation form as :

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

i.e., $AX=B$ … (1)

where $A=\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 1(1-9)-2(-2-3)+1(6+1)=9 \neq 0$

Hence, A^{-1} exists.

Also, $\text{Adj } A = \begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}$$

From (i) we have $X=A^{-1}B$

i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} -1 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ \frac{11}{9} \end{bmatrix}$$

so, $x = -\frac{1}{9}, y = \frac{4}{9}, z = \frac{11}{9}$ is the solution of the given system.

5.6 CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let $AX = B$ be a system of two or three linear equations.

Then, we have the following criteria :

(1) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution, given by $X = A^{-1}B$.

(2) If $|A| = 0$, then the system may or may not be consistent and if consistent, it does not have a unique solution. If in addition,

(a) (Adj A) B $\neq$ O, then the system is inconsistent.

(b) (Adj A) B = O, then the system is consistent and has infinitely many solutions.

Note : These criteria are true for a system of 'n' equations in 'n' variables as well.

We now, verify these with the help of the examples and find their solutions wherever possible.

(a)
$$5x + 7y = 1$$
$$2x - 3y = 3$$

This system is consistent and has a unique solution, because $\frac{5}{2} \neq \frac{7}{-3}$ Here, the matrix equation is $\begin{bmatrix} 5 & 7 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

i.e. $AX = B$ .... (i)

where, $A = \begin{bmatrix} 5 & 7 \\ 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Here, $|A| = 5 \times (-3) - 2 \times 7 = -15 - 14 = -29 \neq 0$

$$\text{and } A^{-1} = \frac{1}{|A|} Adj\ A = \frac{1}{-29}\begin{bmatrix} -3 & -7 \\ -2 & 5 \end{bmatrix} \qquad \text{.... (ii)}$$

From (i), we have $X = A^{-1}B$

$$\text{i.e., } \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-29}\begin{bmatrix} -3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{24}{29} \\ -\frac{13}{29} \end{bmatrix} \quad \text{[From (i) and (ii)]}$$

Thus, $x = \frac{24}{29}$, and $y = \frac{-13}{29}$ is the unique solution of the given system of equations.

(b)
$$3x + 2y = 7$$
$$6x + 4y = 8$$

This system is incosisntent i.e. it has no solution because $\frac{3}{6} = \frac{2}{4} \neq \frac{7}{8}$

In the matrix form the system can be written as

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

or, $AX = B$.

where $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Here, $|A| = 3 \times 4 - 6 \times 2 = 12 - 12 = 0$

$$\text{Adj } A = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}$$

Also, $(\text{Adj } A)\, B = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -20 \\ -18 \end{bmatrix} \neq O$

Thus, the given system of equations is inconsistent.

(c) $\left.\begin{aligned} 3x - y &= 7 \\ 9x - 3y &= 21 \end{aligned}\right\}$ This system is consistent and has infinitely many solutions, because $\dfrac{3}{9} = \dfrac{-1}{-3} = \dfrac{7}{21}$

In the matrix form the system can be written as

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

or, $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}; \; X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 9 & -3 \end{vmatrix} = 3 \times (-3) - 9 \times (-1)$

$$= -9 + 9 = 0$$

$$\text{Adj A} = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}$$

Also, $(\text{Adj A})B = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = O$

$\therefore$ The given system has an infinite number of solutions.

Till now, we have seen that

(i) if $|A| \neq 0$ and $(\text{Adj A})\,B \neq O$, then the system of equations has a non-zero unique solution.

(ii) if $|A| \neq 0$ and $(\text{Adj A})\,B = O$, then the system of equations has only trivial solution $x = y = z = 0$.

(iii) if $|A| = 0$ and $(\text{Adj A})\,B = O$, then the system of equations has infinitely many solutions.

(iv) if $|A| = 0$ and $(\text{Adj A})\,B \neq O$, then the system of equations is inconsistent.

Let us now consider another system of linear equations, where $|A| = 0$ and $(\text{Adj A})\,B \neq O$.
Consider the following system of equations

$$x + 2y + z = 5$$

$$2x + y + 2z = -1$$

$$x - 3y + z = 6$$

In matrix equation form, the above system of equations can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

i.e., $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0 \qquad (\because C_1 = C_3)$

Also, $(\text{Adj A})\,B = \begin{bmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$ [Verify (Adj A) yourself]

$$= \begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \neq O$$

Since $|A| = 0$ and $(\text{Adj A})\,B \neq O$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}(Adj\ \ A)B$$

$$= \frac{\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix}}{0} \text{ which is undefined.}$$

The given system of linear equation will have no solution.

Thus, we find that if $|A| = 0$ and (Adj A) $B \neq O$ then the system of equations will have no solution.

We can summarise the above finding as:

(i) If $|A| \neq 0$ and (Adj A) $B \neq O$ then the system of equations will have a non-zero, unique solution.

(ii) If $|A| \neq 0$ and (Adj A) $B = O$, then the system of equations will have trivial solutions.

(iii) If $|A| = 0$ and (Adj A) $B = O$, then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and (Adj A) $B \neq O$, then the system of equations will have no solution (inconsistent).

Example 5.15 Use matrix inversion method to solve the system of equations:

(i) $$\begin{aligned} 6x + 4y &= 2 \\ 9x + 6y &= 3 \end{aligned}$$

(ii) $$\begin{aligned} 2x - y + 3z &= 1 \\ x + 2y - 3z &= 2 \\ 5y - 5z &= 3 \end{aligned}$$

Solution: (i) The given system in the matrix equation form is

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

i.e., $AX = B$

where, $A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 6 \times 6 - 9 \times 4 = 36 - 36 = 0$

$\therefore$ The system has either infinitely solutions or no solution.

Let $x = k$, then $6k+4y=2$ gives $y = \dfrac{1-3k}{2}$

Putting these values of x and y in the second equation, we have

$$9k+6\left(\frac{1-3k}{2}\right)=3$$

$$\Rightarrow 18k+6-18k=6$$

$$\Rightarrow 6=6, \text{ which is true.}$$

$\therefore$ The given system has infinitely many solutions. These are

$$x=k,\ y=\frac{1-3k}{2}, \text{ where } k \text{ is any arbitrary number.}$$

(ii) The given equations are

$$2x-y+3z=1 \qquad \cdots(1)$$
$$x+2y-z=2 \qquad \cdots(2)$$
$$5y-5z=3 \qquad \cdots(3)$$

In matrix equation form, the given system of equations is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e., $AX=B$

where, $A=\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Now, $|A|=\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{vmatrix}=2\times\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix}+1\times\begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix}+3\times\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$

$$=2(-10+5)+1(-5-0)+3(5-0)$$

$$=-10-5+15=0$$

$\therefore$ The system has either infinitely many solutions or no solution

Let $z = k$. Then from (1), we have $2x - y = 1 - 3k$; and from (2), we have $x + 2y = 2 + k$

Now, we have a system of two equations, namely

$$2x - y = 1 - 3k$$
$$x + 2y = 2 + k$$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-3k \\ 2+k \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1-3k \\ 2+k \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5 \neq 0$

$\therefore$ A^{-1} exists.

Here, $A^{-1} = \dfrac{1}{|A|} \text{ Adj } A = \dfrac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$\therefore$ The solution is $X = A^{-1}B$

$$= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1-3k \\ 2+k \end{bmatrix} = \begin{bmatrix} -k + \frac{4}{5} \\ k + \frac{3}{5} \end{bmatrix}$$

$\therefore$ $x = -k + \dfrac{4}{5}$, $y = k + \dfrac{3}{5}$, where k is any number.

Putting these values of x, y and z in (3), we get

$$5\left(k + \frac{3}{5}\right) - 5(k) = 3$$

$\Rightarrow$ $5k + 3 - 5k = 3 \Rightarrow 3 = 3$, which is true.

∴ The given system of equations has infinitely many solutions, given by

$$x = -k + \frac{4}{5},\ y = k + \frac{3}{5} \text{ and } z = k, \text{ where } k \text{ is any number.}$$

5.7 HOMOGENEOUS SYSTEM OF EQUATIONS

A system of linear equations $AX = B$ with matrix, $B = O$, a null matrix, is called homogeneous system of equations.

Following are some systems of homogeneous equations:

(i) $\begin{aligned} x + 2y &= 0 \\ -2x + 3y &= 0 \end{aligned}$ (ii) $\begin{aligned} 2x + 5y - 3z &= 0 \\ x - 2y + z &= 0 \\ 3x - y - 6z &= 0 \end{aligned}$ (iii) $\begin{aligned} 2x + y - 3z &= 0 \\ x - 2y + z &= 0 \\ 3x - y - 2z &= 0 \end{aligned}$

Let us now solve a system of equations mentioned in (ii).

Given system is

$$\begin{aligned} 2x + 5y - 3z &= 0 \\ x - 2y + z &= 0 \\ 3x - y - 6z &= 0 \end{aligned}$$

In matrix equation form, the system (ii) can be written as

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $AX = O$

where $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $|A| = 2(12 + 1) - 5(-6 - 3) - 3(-1 + 6)$

$$= 26 + 45 - 15$$

$$= 56 \neq 0$$

But $\quad B = O \Rightarrow (\text{Adj } A)\, B = O.$

Thus, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}\ (Adj\ A)\ B$

$= O.$

$\therefore \quad x = 0, y = 0$ and $z = 0.$

i.e., the system of equations will have trivial solution.

Remarks: For a homogeneous system of linear equations, if $|A| \neq 0$ and $(\text{Adj } A)\, B = O$. There will be only trivial solution.

Now, cousider the system of equations mentioned in (iii):

$$2x + y - 3z = 0$$
$$x - 2y + z = 0$$
$$3x - y - 2z = 0$$

In matrix equation form, the above system (iii) can be written as

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $\quad AX = 0$

where, $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} = 2(4+1) - 1(-2-3) - 3(-1+6)$
$$= 10 + 5 - 15$$
$$= 0$$

Also, $B = O \Rightarrow (\text{Adj } A)\, B = O$

Thus, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}\ (\text{Adj } A)\, B = \frac{1}{0}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ The system of equations will have infinitely many solutions which will be non-trivial. Considering the first two equations, we get

$$2x + y = 3z$$
$$x - 2y = -z$$

Solving, we get $x = z,\ y = z$, let $z = k$, where k is any number.

Then x = k, y = k and z = k are the solutions of this system.

For a system of homogenous equations, if $|A| = 0$ and, $(\text{Adj}A)\,B = \text{O}$, there will be infinitely many solutions.

CHECK YOUR PROGRESS 5.4

1. Solve the following system of equations, using the matrix inversion method:

(a) $2x + 3y = 4$, $x - 2y = 5$

(b) $x + y = 7$, $3x - 7y = 11$

(c) $3x + 4y - 5 = 0$, $x - 2y + 6 = 0$

(d) $2x - 3y + 6 = 0$, $6x + y - 8 = 0$

2. Solve the following system of equations using matrix inversion method:

(a) $x + 2y + z = 3$, $2x - y + 3z = 5$, $x + y - z = 7$

(b) $2x + 3y + z = 13$, $3x + 2y - z = 12$, $x + y + 2z = 5$

(c) $-x + 2y + 5z = 2$, $2x - 3y + z = 15$, $-x + y + z = -3$

(d) $2x + y - z = 2$, $x + 2y - 3z = -1$, $5x - y - 2z = -1$

3. Solve the following system of equations, using matrix inversion method:

(a) $x+y+z=0$, $2x-y+z=0$, $x-2y+3z=0$

(b) $3x-2y+3z=0$, $2x+y-z=0$, $4x-3y+2z=0$

(c) $x+y+1=0$, $y+z-1=0$, $z+x=0$

4. Determine whether the following system of equations are consistent or not. If consistent, find the solution:

(a) $2x-3y=5$, $x+y=7$

(b) $2x-3y=5$, $4x-6y=10$

(c) $3x+y+2z=3$, $-2y-z=7$, $x+15y+3z=11$

(d) $x+2y-3z=0$, $4x-y+2z=0$, $3x+5y-4z=0$

LET US SUM UP

- A square matrix is said to be non-singular if its corresponding determinant is non-zero.
- The determinant of the matrix A obtained by deleting the i^th^ row and j^th^ column of A, is called the minor of a_{ij}. It is usually denoted by M_{ij}.
- The cofactor of a_{ij} is defined as $C_{ij}=(-1)^{i+j}M_{ij}$
- Adjoint of a matrix A is the transpose of the matrix whose elements are the cofactors of the elements of the determinat of given matrix. It is usually denoted by Adj A.
- If A is any square matrix of order n, then

  $A\,(\text{Adj}\,A)=(\text{Adj}\,A)\,A=|A|\;I_n$ where I_n is the unit matrix of order n.
- For a given non-singular square matrix A, if there exists a non-singular square matrix B such that $AB=BA=I$, then B is called the multiplicative inverse of A. It is written as $B=A^{-1}$.
- Only non-singular square matrices have multiplicative inverse.
- If $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$, then we can express the system in the matrix equation form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, if $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$X=A^{-1}B=\frac{1}{a_1b_2-a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- A system of equations, given by $AX=B$, is said to be consistent and has a unique solution, if $|A|\neq 0$.
- A system of equations, given by $AX=B$, is said to be inconsistent, if $|A|=0$ and (Adj A) $B\neq$ O.
- A system of equations, given by $AX=B$, is said be be consistent and has infinitely many solutions, if $|A|=0$, and (Adj A) $B=$ O.
- A system of equations, given by $AX=B$, is said to be homogenous, if B is the null matrix.
- A homogenous system of linear equations, $AX=0$ has only a trivial solution $x_1=x_2=\ldots=x_n=0$, if $|A|\neq 0$
- A homogenous system of linear equations, $AX=$ O has infinitely many solutions, if $|A|=0$

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find $|A|$, if

(a) $A=\begin{bmatrix} 1 & 2 & 3 \\ -3 & -1 & 0 \\ -2 & 5 & 4 \end{bmatrix}$ (b) $A=\begin{bmatrix} -1 & 3 & 4 \\ 7 & 5 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

2. Find the adjoint of A, if

(a) $A = \begin{bmatrix} -2 & 3 & 7 \\ -1 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$

Also, verify that $A(\text{Adj } A) = |A| I_3 = (\text{Adj } A)A$, for (a) and (b)

3. Find A^{-1}, if exists, when

(a) $\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

Also, verify that $(A')^{-1} = (A^{-1})'$, for (a), (b) and (c)

4. Find the inverse of the matrix A, if

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

5. Solve, using matrix inversion method, the following systems of linear equations

(a) $\begin{aligned} x + 2y &= 4 \\ 2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 6x + 4y &= 2 \\ 9x + 6y &= 3 \end{aligned}$

(c) $\begin{aligned} 2x + y + z &= 1 \\ x - 2y - z &= \frac{3}{2} \\ 3y - 5z &= 9 \end{aligned}$ (d) $\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$

(e) $\begin{aligned} x + y - 2z &= -1 \\ 3x - 2y + z &= 3 \\ 2x + y - z &= 0 \end{aligned}$

6. Solve, using matrix inversion method

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \quad \frac{8}{x} + \frac{9}{y} - \frac{20}{z} = 3$$

7. Find non-trivial solution of the following system of linear equations:

$$\begin{aligned} 3x+2y+7z &= 0 \\ 4x-3y-2z &= 0 \\ 5x+9y+23z &= 0 \end{aligned}$$

8. Solve the following homogeneous equations :

(a) $$\begin{aligned} x+y-z &= 0 \\ x-2y+z &= 0 \\ 3x+6y-5z &= 0 \end{aligned}$$

(b) $$\begin{aligned} x+2y-2z &= 0 \\ 2x+y-3z &= 0 \\ 5x+4y-9z &= 0 \end{aligned}$$

9. Find the value of 'p' for which the equations

$$\begin{aligned} x+2y+z &= px \\ 2x+y+z &= py \\ x+y+2z &= pz \end{aligned}$$

have a non-trivial solution

10. Find the value of λ for which the following system of equation becomes consistent

$$\begin{aligned} 2x-3y+4 &= 0 \\ 5x-2y-1 &= 0 \\ 21x-8y+\lambda &= 0 \end{aligned}$$

ANSWERS

CHECK YOUR PROGRESS 5.1

1. (a) -12 (b) 10 2. (a) singular (b) non-singular

3. (a) $M_{11} = 4; M_{12} = 7; M_{21} = -1; M_{22} = 3$

   (b) $M_{11} = 5; M_{12} = 2; M_{21} = 6; M_{22} = 0$

4. (a) $M_{21} = 11; M_{22} = 7; M_{23} = 1$

   (b) $M_{31} = -13; M_{32} = -13; M_{33} = 13$

5. (a) $C_{11} = 7; C_{12} = -9; C_{21} = 2; C_{22} = 3$

   (b) $C_{11} = 6; C_{12} = 5; C_{21} = -4; C_{22} = 0$

6. (a) $C_{21} = 1; C_{22} = -8; C_{23} = -2$

   (b) $C_{11} = -6; C_{12} = 10; C_{33} = 2$

CHECK YOUR PROGRESS 5.2

1. (a) $\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

CHECK YOUR PROGRESS 5.3

1. (a) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ -\frac{8}{5} & \frac{6}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{17}{24} & -\frac{1}{3} & -\frac{11}{24} \end{bmatrix}$

4. $(A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

CHECK YOUR PROGRESS 5.4

1. (a) $x = \frac{23}{7},\ y = \frac{-6}{7}$

   (b) $x = 6,\ y = 1$

   (c) $x = -\frac{7}{5},\ y = \frac{23}{10}$

   (d) $x = \frac{27}{30},\ y = \frac{13}{5}$

2. (a) $x = \frac{58}{11},\ y = -\frac{2}{11},\ z = -\frac{21}{11}$

   (b) $x = 2,\ y = 3,\ z = 0$

   (c) $x = 2,\ y = -3,\ z = 2$

   (d) $x = 1,\ y = 2,\ z = 2$

3. (a) $x = 0,\ y = 0,\ z = 0$

   (b) $x = 0,\ y = 0,\ z = 0$

   (c) $x = 0,\ y = 0,\ z = 0$

4. (a) Consistent; $x = \frac{26}{5},\ y = \frac{9}{5}$

   (b) Consistent; infinitely many solutions

   (c) Inconsistent

   (d) Trivial solution, $x = y = z = 0$

TERMINAL EXERCISE

1. (a) -31

   (b) -24

2. (a) $\begin{bmatrix} 4 & -3 & -13 \\ -4 & 5 & 3 \\ 4 & -3 & -5 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 8 & -7 \\ -13 & 13 & 13 \\ 5 & 1 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} -\frac{1}{18} & \frac{1}{6} \\ \frac{7}{36} & -\frac{1}{12} \end{bmatrix}$

   (b) $\begin{bmatrix} \frac{5}{7} & -\frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}$

   (c) $\begin{bmatrix} -\frac{1}{7} & -\frac{5}{14} \\ -\frac{2}{7} & -\frac{3}{14} \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 \\ 3 & \frac{2}{3} & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

5. (a) $x = 2,\ y = 1$

(b) $x = k,\ y = \frac{1}{2} - \frac{3}{2}k$

(c) $x = 1,\ y = \frac{1}{2},\ z = -\frac{3}{2}$

(d) $x = 2,\ y = -1,\ z = 1$

(e) $x = \frac{1}{2},\ y = -\frac{1}{2},\ z = \frac{1}{2}$

6. $x = 2,\ y = 3,\ z = 5$

7. $x = -k,\ y = -2k,\ z = k$

8. (a) $x = k,\ y = 2k,\ z = 3k$

(b) $x = y = z = k$

9. $p = 1, -1, 4$

10. $\lambda = -5$

6

LINEAR INEQUATIONS AND THEIR APPLICATIONS

You have already read about linear and quadratic equations in one or two variables. You have also learnt the methods of solving the linear and quadratic equations and have applied this knowledge in solving word problems from daily life.

Some times, it is not possible to translate a word problem in the form of an equation. Let us consider the following situation:

Alok goes to market with Rs. 30 to buy pencils. The cost of one pencil is Rs. 2.60. If x denotes the number of pencils which he buys, then he will spend an amount of Rs. 2.60x. This amount cannot be equal to Rs. 30 as x is a natural number. Thus.

$$2.60\,x < 30 \qquad \ldots \text{(i)}$$

Let us consider one more situation where a person wants to buy chairs and tables with Rs. 50,000 in hand. A table costs Rs. 550 while a chair costs Rs. 450. Let x be the number of chairs and y be the number of tables he buys, then his total cost = Rs.$(550\,x + 450\,y)$

Thus, in this case we can write, $550x + 450y \leq 50{,}000$

or $\quad 11x + 9y \leq 1000 \qquad \ldots \text{(ii)}$

Statement (i) involves the sign of inequality '<' and statement (ii) consists of two statements: $11x+9y < 1000$, $11x+9y = 1000$ in which the first one is not an equation: Such statements are called Inequations.

In this lesson, we will discuss linear inequations and solve problems related to diet, business and transportation using graphical method.

OBJECTIVES

After studying this lesson, you will be able to:

- differentiate between a linear equation and a linear inequation;
- state that a planar region represents the solution of a linear inequation;
- represent graphically a linear inequation in two variables;
- show the solution of an inequation by shading the appropriate region;
- solve graphically a system of two or three linear inequations in two variables;
- define the terms such as objective function, constraints, feasible region, optimisation;
- formulate problems related to diet, business, transportation, etc; and
- solve above problems using graphical method.

EXPECTED BACKGROUND KNOWLEDGE

- Solution of linear equations in one or two variables.
- Graph of linear equations in one or two variables in a plane.
- Graphical solution of a system of linear equations in two variables.

6.1 INEQUATION

In this lesson we will discuss more about linear inequations and their applications from daily life. A statement involving a sign of equality (=) is an equation.

Similarly, a statement involving a sign of inequality, <, >, ≤, or ≥ is called an inequation. Some examples of inequations are:

(i) $2x + 5 > 0$

(ii) $3x - 7 < 0$

(iii) $ax + b \geq 0,\ a \neq 0$

(iv) $ax + b \leq c,\ a \neq 0$

(v) $3x + 4y \leq 12$

(vi) $x^2 - 5x + 6 < 0$

(vii) $ax + by + c \geq 0$

(v) and (vii) are inequations in two variables and all other inequations are in one variable. (i) to (v) and (vii) are linear inequations and (vi) is a quadratic inequation.

In this lesson, we shall study about linear inequations in one or two variables only.

6.2 SOLUTIONS OF LINEAR INEQUATIONS IN ONE/TWO VARIABLES

Solving an inequation means to find the value (or values) of the variable (s), which when substituted in the inequation, satisfies it.

For example, for the inequation $2.60x<30$ (statement) (i) all values of $x \leq 11$ are the solutions. (x is a whole number)

For the inequation $2x + 16 > 0$, where x is a real number, all values of x which are > -8 are the solutions.

For the linear inequation in two variables, like $ax + by + c \geq 0$, we shall have to find the pairs of values of x and y which make the given inequation true.

Let us consider the following situation :

Anil has Rs. 60 and wants to buy pens and pencils from a shop. The cost of a pen is Rs. 5 and that of a pencil is Rs. 3 If x denotes the number of pens and y, the number of pencils which Anil buys, then we have the inequality $5x + 3y \leq 60$ ... (i)

Here, $x = 6, y = 10$ is one of the solutions of the inequation (i). Similarly $x = 5, y = 11; x = 4, y = 13; x = 10, y = 3$ are some more solutions of the inequation.

In solving inequations, we follow the rules which are as follows :

1. Equal numbers may be added (or subtracted) on both sides of an inequation.

   Thus (i) if $a > b$ then $a + c > b + c$ and $a - c > b - c$

   and (ii) if $a \leq b$ then $a + d \leq b + d$ and $a - d \leq b - d$

2. Both sides of an inequation can be multiplied (or divided) by the same positive number.

   Thus (i) if $a > b$ and $c > 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

   and (ii) if $a \leq b$ and $c > 0$ then $ac \leq bc$ and $\frac{a}{c} \leq \frac{b}{c}$

3. When both sides of an inequation are multiplied by the same negative number, the sign of inequality gets reversed.

   Thus (i) if $a > b$ and $d < 0$ then $ad < bd$ and $\frac{a}{d} < \frac{b}{d}$

   and (ii) if $a \leq b$ and $c < 0$ then $ac \geq bc$ and $\frac{a}{c} \geq \frac{b}{c}$

6.3 GRAPHICAL REPRESENTATION OF LINEAR INEQUATIONS IN ONE OR TWO VARIABLES.

In Section 6.2, while translating word problem of purchasing pens and pencils, we obtained the following linear inequation in two variables x and y:

$5x + 3y \leq 60$ .... (i)

Let us now find all solutions of this inequation, keeping in mind that x and y here can be only whole numbers.

To start with, let $x = 0$.

Thus, we have $3y \le 60$ or $y \le 20$, i,e the values of y corresponding to $x = 0$ can be 0,1,2,3............, 20 only Thus, the solutions with $x = 0$ are

(0,0), (0,1), (0,2) ....(0, 20)

Similarly the other solutions of the inequation, when $x = 1,2, \ldots 12$ are

(1,0) (1,1) (1,2) .... (1,18)
(2,0) (2,1) (2,2) .... (2,16)
....
....
(10,0) (10,1) (10,2), (10,3)
(11,0) (11,1)
(12,0)

You may note that out of the above ordered pairs, some pairs such as (0,20), (3, 15), (6, 10), (9, 5), (12,0) satisfy the equation $5x + 3y = 60$ which is a part of the given inequation and all other possible solutions lie on **one of the two half planes** in which the line $5x + 3y = 60$, divides the xy - plane.

If we now extend the domain of x and y from whole numbers to real numbers, the inequation $5x + 3y \le 60$ will represent one of the two half planes in which the line $5x + 3y = 60$, divides the xy-plane.

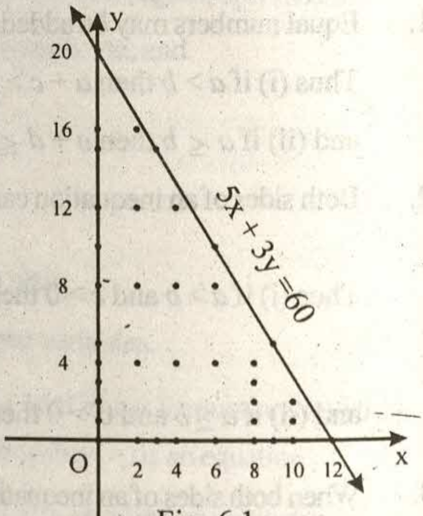


Fig. 6.1

Thus we can generalize as follows :

If a, b, c, are real numbers, then $ax + by + c = 0$ is called a linear equation in two variables x and y, where as $ax + by + c \le 0$ or $ax + by + c \ge 0$, $ax + by + c > 0$ and $ax + by + c < 0$ are called linear inequations in two variables x and y.

The equation $ax + by + c = 0$ is a straight line which divides the xy plane into two half planes which are represented by $ax + by + c \ge 0$ and $ax + by + c \le 0$.

For example $3x + 4y - 12 = 0$ can be represented by line AB, in the xy - plane as shown in Fig. 6.2

The line AB divides the cordinate plane into two half - plane regions :

(i) half plane region I above the line AB

(ii) half plane region II below the line AB. One of the above region represents the inequality $3x + 4y - 12 \le 0$ ...(i) and the other region will be represented by $3x + 4y - 12 \ge 0$. ... (ii)

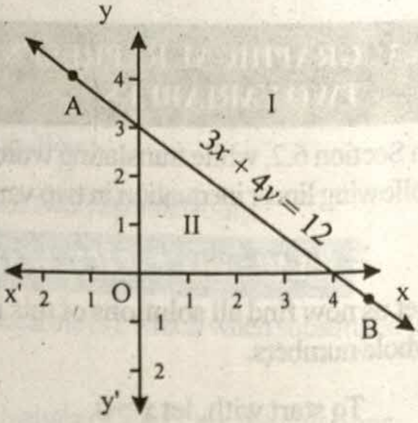


Fig. 6.2

To identify the half plane represented by inequation (i), we take any arbitrary point, preferably origin, if it does not lie on AB. If the point satisfies the inequation (i), then the half plane in which the arbitrary point lies, is the desired half plane. In this case, taking origin as the arbitrary point we have

$0+0-12 \leq 0$ i.e $-12 \leq 0$. Thus origin satisfies the inequation $3x+4y-12 \leq 0$. Now, origin lies in half plane region II. Hence the inequalty $3x+4y-12 \leq 0$ represents half plane II and the inequality $3x+4y-12 \geq 0$ will represent the half plane I

Example 6.1 Show on graph the region represented by the inequation $x+2y \geq 5$.

Solution : The given inequation is $x+2y \geq 5$

Let us first take the corresponding linear equation $x+2y=5$ and draw its graph with the help of the following table :

x	1	3	5
y	2	1	0

Since (0,0) does not lie on the line AB, so we can select (0,0) as the arbitrary point. Since $0+0 \geq 5$ is not true

∴ The desired half plane is one, in which origin does not lie

∴ The desired half plane is the shaded one (See Fig. 6.3)

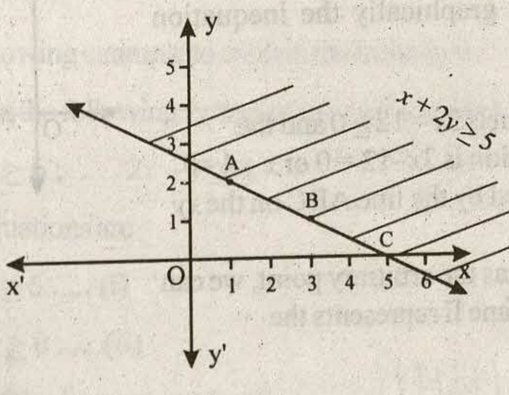


Fig. 6.3

Before taking more examples, it is important to define the following :

(i) **Closed Half Plane:** A half plane is said to be closed half plane if all points on the line separating the two half planes are also included in the solution of the inequation. The Half plane in Example 6.1 is a closed half plane.

(ii) **An Open Half Plane :** A half plane in the xy plane is said to be an open half plane if the points on the line separting the planes are not included in the half plane.

Example 6.2 Draw the graph of inequation $x-5y>0$

Solution : The given inequation is $x-5y>0$

The corresponding linear equation is $x-5y=0$ we have the following table.

x	0	5	–5
y	0	1	–1

The line AOB divides xy - plane into two half planes I and II. As the line AOB passes through origin, we consider any other arbitrary point (say) P (3,4) which is in half plane I. Let us see whether it satisfies the given inequation $x-5y>0$

$\therefore$ Then $3-5(4)>0$ or $-20>0$, or $-17>0$ which is not true

$\therefore$ The desired half plane is II

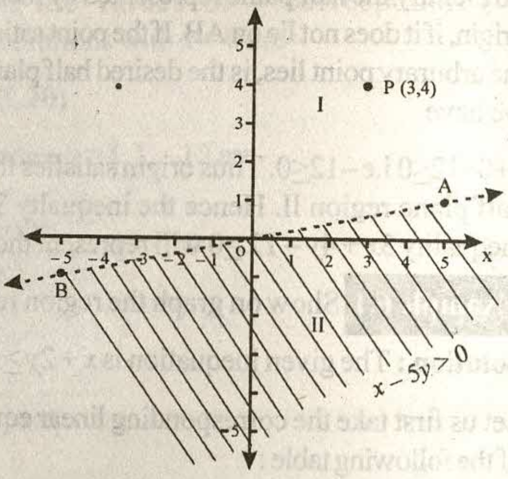


Fig. 6.4

Again the inequation is a **strict** inequation $x-5y>0$

$\therefore$ Line AOB is not a part of the graph and hence has been shown as a dotted line.

Hence, the graph of the given inequation is the shaded region half plane II excluding the line AOB.

Example 6.3 Represent graphically the inequation $3x-12\geq 0$

Solution : Given inequation is $3x-12\geq 0$ and the corresponding linear equation is $3x-12=0$ or $x-4=0$ or $x=4$ which is represented by the line ABC on the xy - plane (See Fig. 6.5). Taking (0,0) as the arbitrary point, we can say that $0 \ngeq 4$ and so, half plane II represents the inequation $3x-12\geq 0$

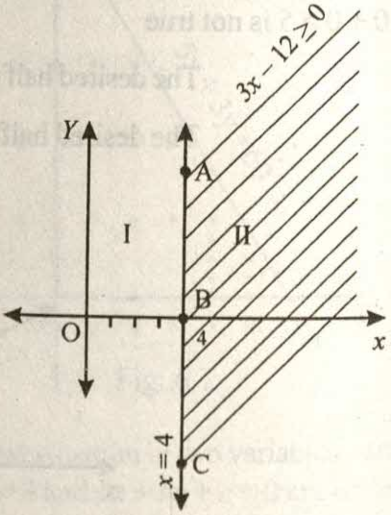


Fig. 6.5

Example 6.4 Solve graphically the inequation $2y+4\geq 0$

Solution : Here the inequation is $2y+4\geq 0$ and the corresponding equation is $2y+4=0$ or $y=-2$

The line ABC represents the line $y=-2$ which divides the xy - plane into two half planes and the inequation $2y+4\geq 0$ is represented by the half plane I.

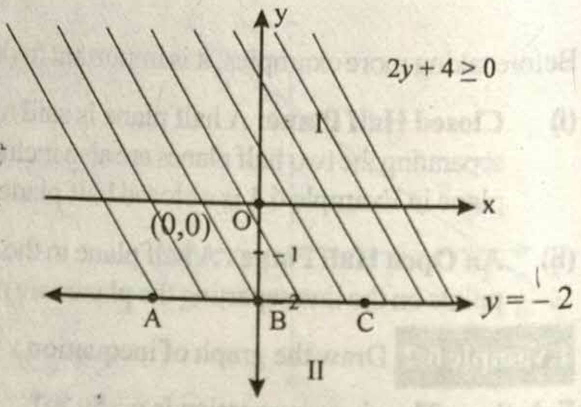


Fig. 6.6

CHECK YOUR PROGRESS 6.1

Represent the solution of each of the following inequations graphically in two dimensional plane:

1. $2x+y\geq 8$
2. $x-2y\leq 0$
3. $3x+6\geq 0$
4. $8-2y\geq 2$
5. $3y\geq 6-2x$
6. $3x\geq 0$
7. $y\leq 4$
8. $y>2x-8$
9. $-y<x-5$
10. $2y\leq 8-4x$

6.4 GRAPHICAL SOLUTION OF A SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES.

You already know how to solve a system of linear equations in two variables.

Now, you have also learnt how to solve **linear inequations** in two variables graphically. We will now discuss the technique of finding the solutions of a system of simultaneous linear inequations. By the term solution of a system of simultaneons linear inequations we mean, finding all ordered pairs (x,y) for which each linear inequation of the system is satisfied.

A system of simultaneous inequations may have no solution or an infinite number of solutions represented by the region bounded or unbounded by straight lines corresponding to linear inequations.

We take the following example to explain the technique.

Example 6.5 Solve the following system of inequations graphically:

$$x+y\geq 6\,; \qquad 2x-y\geq 0.$$

Solution : Given inequations are

$$x+y\geq 6 \text{ ..... (i)}$$

and $$2x-y\geq 0 \text{ ..... (ii)}$$

We draw the graphs of the lines $x+y=6$ and $2x-y=0$ (Fig. 6.7)

The inequation (i) represent the shaded region above the line $x+y=6$ and inequations (ii) represents the region on the right of the line $2x-y=0$

The common region represented by the double shade in Fig. 6.7 represents the solution of the given system of linear inequations.

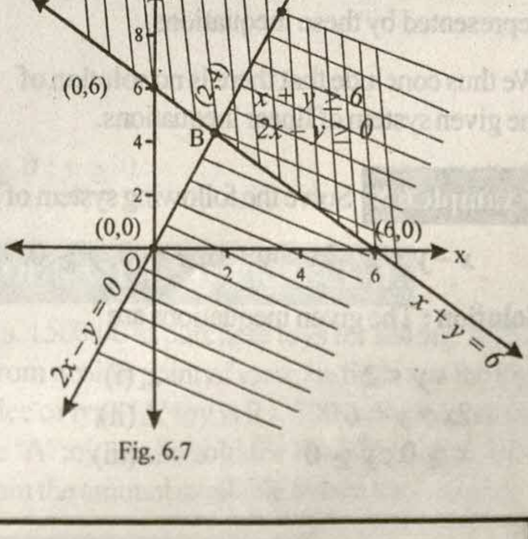


Fig. 6.7

Example 6.6 Find graphically the solution of the following system of linear inequations :

$$x + y \le 5, \quad 4x + y \ge 4, \quad x + 5y \ge 5, \quad x \le 4, \; y \le 3.$$

Solution : Given inequations are

$x + y \le 5$ ..... (i)

$4x + y \ge 4$ ..... (ii)

$x + 5y \ge 5$ ..... (iii)

$x \le 4$ ..... (iv)

and $y \le 3$ ..... (v)

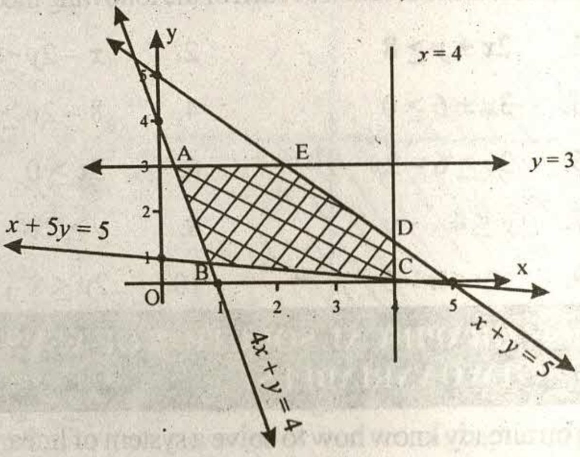


Fig. 6.8

We draw the graphs of the lines $x + y = 5$, $4x + y = 4$, $x + 5y = 5$, $x = 4$ and $y = 3$ (Fig. 6.8)

The inequation (i) represents the region below the line $x + y = 5$. The inequations (ii) represents the region on the right of equation $4x + y = 4$ and the region above the line $x + 5y = 5$ represents the inequation (iii). Similarly after shading the regions for inequations (iv) and (v) we get the common region as the bounded region ABCDE as shown in (Fig. 6.8) The co-ordinates of the points of the shaded region satisfy the given system of inequations and, therefore all these points represent solution of the given system.

Example 6.7 Solve graphically the following system of inequations :

$$x + 2y \le 3, \; 3x + 4y \ge 12, \; x \ge 0, y \ge 0.$$

Solution : We represent the inequations $x + 2y \le 3, 3x + 4y \ge 12, x \ge 0, y \ge 0$ by shading the corresponding regions on the graph as shown in Fig. 6.9

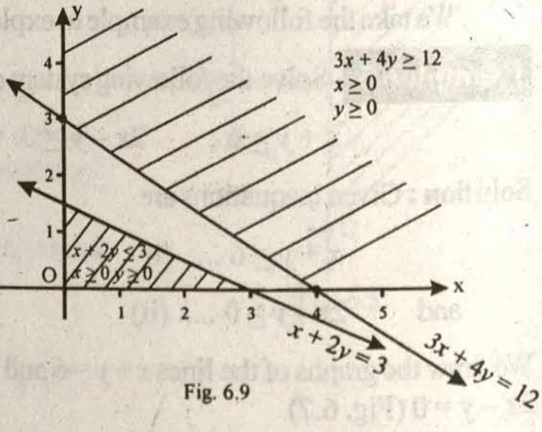


Fig. 6.9

Here we find that there is no common region represented by these inequations.

We thus conclude that there is no solution of the given system of linear inequations.

Example 6.8 Solve the following system of linear inequations graphically :

$$x - y < 2, \; 2x + y < 6; \; x \ge 0, y \ge 0.$$

Solution : The given inequations are

$x - y < 2$ ... (i)

$2x + y < 6$ ... (ii)

$x \ge 0 \,; y \ge 0$ ... (iii)

After representing the inequations

$x - y < 2$, $2x + y < 6$, $x \geq 0$ and $y \geq 0$ on the graph we find the common region which is the bounded region $OABC$ as shown in Fig. 6.10

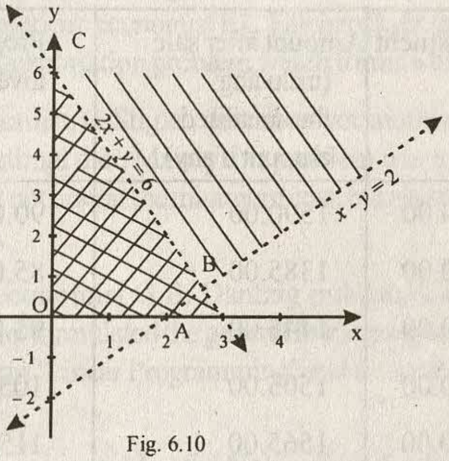


Fig. 6.10

CHECK YOUR PROGRESS 6.2

Solve each of the follwing systems of linear inequations in two variables graphically :

1. $x \geq 3,\ y \geq 1.$

2. $y \geq 2x, y \leq 2.$

3. $2x + y - 3 \geq 0, x - 2y + 1 \leq 0.$

4. $3x + 4y \leq 12,\ 4x+3y \leq 12,\ x \geq 0,\ y \geq 0$

5. $2x + 3y \geq 3,\ 3x + 4y \leq 18,\ 7x - 4y + 14 \geq 0,\ x - 6y \leq 3,\ x \geq 0, y \geq 0$

6. $x + y \geq 9, 3x + y \geq 12, x \geq 0,\ y \geq 0$

7. $x + y \geq 1; 2x + 3y \leq 6, x \geq 0, y \geq 0.$

8. $x + 3y \geq 10; x + 2y \leq 3, x - 2y \leq 2, x \geq 0\ ; y \geq 0$

6.5 INTRODUCTION TO LINEAR PROGRAMMING PROBLEM

A toy dealer goes to the wholesale market with Rs. 1500.00 to purchase toys for selling. In the market there are various types of toys available. From quality point of view, he finds that the toy of type 'A' and type 'B' are suitable. The cost price of type 'A' toy is Rs. 300 each and that of type 'B' is Rs. 250 each. He knows that the type 'A' toy can be sold for Rs. 325 each, while the type 'B' toy can be sold for Rs. 265 each. Within the amount available to him he would like to make maximum profit. His problem is to find out how many type 'A' and type 'B' toys should be purchased so as to get the maximum profit.

He can prepare the following table taking into account all possible combinations of type 'A' and type 'B' toys subject to the limitation on the investment.

'A' type	'B' type	Investment	Amount after sale (including the unutilised amount if any)	Profit on the investment
0	6	1500.00	1590.00	90.00
1	4	1300.00	1385.00	85.00
2	3	1350.00	1445.00	95.00
3	2	1400.00	1505.00	105.00
4	1	1450.00	1565.00	115.00
5	0	1500.00	1625.00	125.00

Now, the decision leading to maximum profit is clear. Five type A toys should be purchased.

The above problem was easy to handle because the choice was limited to two types, and the number of items to be purchased was small. Here, all possible combinations were thought of and the corresponding gain calculated. But one must make sure that he has taken all possibilities into account.

A situation faced by a retailer of radio sets similar to the one given above is described below.

A retailer of radio sets wishes to buy a number of transistor radio sets from the wholesaler. There are two types (type A and type B) of radio sets which he can buy. Type A costs Rs.360 each and type *B* costs Rs. 240 each. The retailer can invest up to Rs. 5760. By selling the radio sets, he can make a profit of Rs. 50 on each set of type A and of Rs. 40 on each set of type B. How many of each type should he buy to maximize his total profit?

Here we have to maximize the profit. Sometimes we come across a problem in which the costs are to be minimized. Consider the following problem :

Two tailors A and B earn Rs.150 and Rs.200 per day respectively. A can stitch 6 shirts and 4 pants per day, while B can stitch 4 shirts and 7 pants per day. How many days shall each work if they want to produce at least 60 shirts and 72 pants at a minimum labour cost?

In this problem we have to minimise the labour cost.

These types of problems of maximisation and minimisation are called optimisation problems.

The technique followed by mathematicians to solve such problems is called **'Linear Programming'**.

6.5.1 HISTORICAL BACKGROUND

The technique of linear programming is of recent origin. It started during the second world war, when the war operations had to be planned to economise the expenditure, minimise losses and maximise the damage to the enemy.

The first problem in linear programming was formulated in 1941 by the Russian mathematician L.Kantorovich and the American economist F.L. Hilcheock, both of them worked independently. This is the well known transportation problem which forms a branch of linear programming.

In 1945, an English economist, G.Stigler, described yet another linear programming problem-that of determining an optimal diet. Mainly this problem was to determine the quantities of 77 foods that are to be bought not only at the minimum cost, but also to satisfy minimum requirements of nine nutritive elements.

In 1947, an American economist G.B. Dantzig published a paper in the famous journal 'Econometrics' wherein he formulated the general linear programming problem. Dantzig is also credited with using the term 'Linear Programming' and for solution of the problem by analytical methods.

In 1974, L. Kantorovich was awarded the Noble Prize for Economis for his work on these problems together with another famous American mathematical-economist T.C. Koopmans.

6.6 DEFINITIONS OF VARIOUS TERMS INVOLVED IN LINEAR PROGRAMMING

A close examination of the examples cited in the introduction points out one basic property that all these problems have in common, i.e., in each example, we were concerned with maximising or minimising some quantity.

In Examples 1 and 2, we wanted to maximise the return on the investment. In Example 3, we wanted to minimise the labour cost. In linear programming terminology the maximization or minimization of a quantity is referred to as the objective of the problem.

6. 6.1 OBJECTIVE FUNCTION

In a linear programming problem. z, the linear function of the variables which is to be optimized is called **objective function.**

Here, a linear form means a mathematical expression of the type

$$a_1x_1 + a_2x_2 + \ldots.. + a_nx_n,$$

where $a_1, a_{2, \ldots\ldots}, a_n$ are **constants** and $x_1, x_{2, \ldots\ldots}, x_n$ are **variables.**

In linear programming problems, the products, services, projects etc. that are competing with each other for sharing the given limited resources are called the variables or decision variables.

6. 6.2 CONSTRAINTS

The limitations on resources (like cash in hand, production capacity, man power, time, machines, etc.) which are to be allocated among various competing variables are in the form of linear equations or inequations (inequalities) and are called **constraints or restrictions.**

6. 6.3 NON-NEGATIVE RESTRICTIONS

All decision variables must assume non-negative values, as negative values of physical quantities is an impossible situation.

6.7 FORMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of a linear programming problem as a mathematical model involves the following key steps.

Step 1 : Identify the decision variables to be determined and express them in terms of algebraic symbols such as $x_{1,} x_{2,} x_{3,}$ ....

Step 2 : Identify all the limitations in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 3 : Identify the objective which is to be optimised (maximised or minimised) and express it as a linear function of the above defined decision variables.

Example 6.9 A retailer wishes to buy a number of transistor radio sets of types *A* and *B*. Type *A* cost Rs.360 each and type *B* cost Rs. 240 each. The retailer knows that he cannot sell more than 20 sets, so he does not want to buy more than 20 sets and he cannot afford to pay more than Rs.5760. His expectation is that he would get a profit of Rs.50 for each set of type *A* and Rs.40 for each set of type *B*. Form a mathematical model to find how many of each type should be purchased in order to make his total profit as large as possible?

Solution : Suppose the retailer purchases x_1 sets of type *A* and x_2 sets of type *B*. Since the number of sets of each type is non-negative, so we have

$$x_1 \geq 0, \qquad \cdots(1)$$

$$x_2 \geq 0, \qquad \cdots(2)$$

Also the cost of x_1 sets of type *A* and x_2 sets of type *B* is 360 x_1 + 240 x_2 and it should be equal to or less than Rs.5760, that is,

$$360x_1 + 240x_2 \leq 5760$$

or $\quad 3x_1 + 2x_2 \leq 48 \qquad \cdots(3)$

Further, the number of sets of both types should not exceed 20, so

$$x_1 + x_2 \leq 20 \qquad \cdots(4)$$

Since the total profit consists of profit derived from selling the x_1 type *A* sets and x_2 type *B* sets, therefore, the retailer earns a profit of Rs.50 x_1 on type *A* sets and Rs.40 x_2 on type *B* sets. So the total profit is given by :

$$z = 50x_1 + 40x_2 \qquad \cdots(5)$$

Hence, the mathematical formulation of the given linear programming problem is as follows :

Find x_1, x_2 which

Maximise z = $50x_1 + 40x_2$ **(Objective function) subject to the conditions**

$$\left.\begin{array}{l} 3x_1 + 2x_2 \le 48 \\ x_1 + x_2 \le 20 \\ x_1 \ge 0, x_2 \ge 0 \end{array}\right\} \textbf{Constraints}$$

Example 6.10 A soft drink company has two bottling plants, one located at *P* and the other at *Q*. Each plant produces three different soft drinks *A*, *B*, and *C*. The capacities of the two plants in terms of number of bottles per day, are as follows :

Plants → Products ↓	P	Q
A	3000	1000
B	1000	1000
C	2000	6000

A market survey indicates that during the month of May, there will be a demand for 24000 bottles of *A*, 16000 bottles of *B* and 48000 bottles of *C*. The operating cost per day of running plants *P* and *Q* are respectively Rs.6000 and Rs.4000. How many days should the firm run each plant in the month of May so that the production cost is minimised while still meeting the market demand.

Solution : Suppose that the firm runs the plant *P* for x_1 days and plant Q for x_2 days in the month of May in order to meet the market demand.

The per day operating cost of plant *P* is Rs.6000. Therefore, for x_1 days the operating cost will be Rs.6000 x_1.

The per day operating cost of plant *Q* is Rs.4000. Therefore, for x_2 days the operating cost will be Rs.4000 x_2.

Thus the total operating cost of two plants is given by :

$$z = 6000x_1 + 4000x_2 \qquad \cdots(1)$$

Plant P produces 3000 bottles of soft drink A per day. Therefore, in x_1 days plant P will produce 3000 x_1 bottles of soft drink A.

Plant Q produces 1000 bottles of soft drink A per day.

Therefore, in x_2 days plant Q will produce 1000 x_2 bottles of soft drink A.

Total production of soft drink A in the supposed period is $3000\,x_1 + 1000\,x_2$

But there will be a demand for 24000 bottles of this soft drink, so the total production of this soft drink must be greater than or equal to this demand.

$$\therefore \quad 3000x_1 + 1000x_2 \geq 24000$$

$$\text{or} \quad 3x_1 + x_2 \geq 24 \qquad \cdots(2)$$

Similarly, for the other two soft drinks, we have the constraints

$$1000x_1 + 1000x_2 \geq 16000$$

$$\text{or} \quad x_1 + x_2 \geq 16 \qquad \cdots(3)$$

and

$$2000x_1 + 6000x_2 \geq 48000$$

$$\text{or} \quad x_1 + 3x_2 \geq 24 \qquad \cdots(4)$$

x_1 and x_2 are non-negative being the number of days, so

$$x_1 \geq 0,\ x_2 \geq 0 \qquad \cdots(5)$$

Thus our problem is to find x_1 and x_2 which

Minimize $z = 6000x_1 + 4000x_2$ ***(objective function)***

subject to the conditions

$$\left.\begin{array}{l} 3x_1 + x_2 \geq 24 \\ x_1 + x_2 \geq 16 \\ x_1 + 3x_2 \geq 24 \end{array}\right\} \quad \textit{(constraints)}$$

and $x_1 \geq 0,\ x_2 \geq 0$

Example 6.11 A firm manufactures two types of products A and B and sells them at a profit of Rs.2 on type A and Rs.3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and 2 minutes on H, type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hours and

40 minutes while machine *H* is available for 10 hours during one working day. Formulate the problem as a linear programming problem so as to maximise profit.

Solution : Let x_1 be the number of products of type *A* and x_2 be the number of products of type *B*.

The given information in the problem can systematically be arranged in the form of following table :

Machine	Processing time of the products (in minute)		Available time (in minute)
	Type *A* (x_1 units)	Type *B* (x_2 units)	
G	1	1	400
H	2	1	600
Profit per unit	Rs.2	Rs.3	

Since the profit on type *A* is Rs.2 per product, so the profit on selling x_1 units of type *A* will be $2x_1$. Similarly, the profit on selling x_2 units of type *B* will be $3x_2$. Therefore, total profit on selling x_1 units of type *A* and x_2 units of type *B* is given by

$$z = 2x_1 + 3x_2 \quad \text{(objective function)} \quad \cdots(1)$$

Since machine G takes 1 minute time on type *A* and 1 minute time on type *B*, therefore, the total number of minutes required on machine *G* is given by

$$x_1 + x_2$$

But the machine *G* is not available for more than 6 hours and 40 minutes (i.e., 400 minutes). Therefore,

$$x_1 + x_2 \le 400 \quad \cdots(2)$$

Similarly, the total number of minutes required on machine *H* is given by

$$2x_1 + x_2$$

Also, the machine *H* is available for 10 hours (i.e., 600 minutes). Therefore,

$$2x_1 + x_2 \le 600 \quad \cdots(3)$$

Since, it is not possible to produce negative quantities, so

$$x_1 \ge 0,\ x_2 \ge 0 \quad \cdots(4)$$

Thus, the problem is to find x_1 and x_2 which

Maximize $z = 2x_1 + 3x_2$ (objective function)

subject to the conditions

$$x_1 + x_2 \le 400$$
$$2x_1 + x_2 \le 600$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Example 6.12 A furniture manufacturer makes two types of sofas – sofa of type *A* and sofa of type *B*. For simplicity, divide the production process into three distinct operations, say carpentary, finishing and upholstery. The amount of labour required for each operation varies. Manufacture of a sofa of type *A* requires 6 hours of carpentary, 1 hour of finishing and 2 hours of upholstery. Manufacture of a sofa of type *B* requires 3 hours of carpentary, 1 hour of finishing and 6 hours of upholstery. Owing to limited availability of skilled labour as well as of tools and equipment, the factory has available each day 96 man hours of carpentary, 18 man hours for finishing and 72 man hours for upholstery. The profit per sofa of type *A* is Rs.80 and the profit per sofa of type *B* is Rs. 70. How many sofas of type *A* and type *B* should be produced each day in order to maximise the profit? Formulate the problems as linear programming problem.

Solution : The different operations and the availability of man hours for each operation can be put in the following tabular form :

Operations	**Sofa of type *A***	**Sofa of type *B***	**Available labour**
Carpentary	6 hours	3 hours	96 man hours
Finishing	1 hour	1 hour	18 man hours
Upholstery	2 hours	6 hours	72 man hours
Profit	Rs. 80	Rs. 70	

Let x_1 be the number of sofas of type *A* and x_2 be the number of sofas of type *B*.

In order to achieve a large profit, one need only to manufacture a large number of sofas of type *A* and type *B*. But, owing to restricted availability of tools and labour, the factory cannot manufacture an unlimited quantity of furniture.

Each row of the chart gives one restriction. The first row says that the amount of carpentary required is 6 hours for each sofa of type *A* and 3 hours for each sofa of type *B*. Further, only 96 man hours of carpentary are available per day. We can compute the total number of man hours of carpentary required per day to produce x_1 sofas of type *A* and x_2 sofas of type *B* as follows :

Number of man - hours per day of carpentary

= { (Number of hours carpentary per sofa of type *A*) × (Number of sofas of type *A*) }

+ { (Number of hours carpentary per sofa of type *B*) × (Number of sofas of type *B*) }

$= 6x_1 + 3x_2$

The requirement that at most 96 man hours of carpentary per day means

$$6x_1 + 3x_2 \leq 96$$

or $2x_1 + x_2 \leq 32$ ...(1)

Similarly, second and third row of the chart give the restrictions on finishing and upholstery respectively as

$x_1 + x_2 \leq 18$ ...(2)

and

$$2x_1 + 6x_2 \leq 72$$

or $x_1 + 3x_2 \leq 36$ ...(3)

Since, the number of the sofas cannot be negative, therefore

$x_1 \geq 0, x_2 \geq 0$ ...(4)

Now, the profit comes from two sources, that is, sofas of type *A* and sofas of type *B*. Therefore,

Profit = (Profit from sofas of type *A*) + (Profit from sofas of type *B*)

= { (Profit per sofa of type *A*) × (Number of sofas of type *A*) }

+ { (Profit per sofa of type *B*) × (Number of sofas of type *B*) }

$z = 80x_1 + 70x_2$ **(objective function)** ...(5)

Thus, the problem is to find x_1 and x_2 which

Maximize $z = 80x_1 + 70x_2$ (objective function)

subject to the constraints

$$\left.\begin{array}{l} 2x_1 + x_2 \leq 32 \\ x_1 + x_2 \leq 18 \\ x_1 + 3x_2 \leq 36 \\ x_1 \geq 0,\ x_2 \geq 0 \end{array}\right\} \quad \textbf{(Constraints)}$$

CHECK YOUR PROGRESS 6.3

1. A company is producing two products *A* and *B*. Each product is processed on two machines *G* and *H*. Type *A* requires 3 hours of processing time on *G* and 4 hours on *H*; type *B* requires 4 hours of processing time time on G and 5 hours on *H*. The available time is 18 hours and 21 hours for operations on *G* and *H* respectively. The products *A* and *B* can be sold at the profit of Rs. 3 and Rs. 8 per unit respectively. Formulate the problem as a linear programming problem.

2. A furniture dealer deals in only two items, tables and chairs. He has Rs. 5000 to invest and a space to store at most 60 pieces. A table costs him Rs. 250 and a chair Rs. 50. He can sell a table at a profit of Rs. 50 and a chair at a profit of Rs. 15. Assuming, he can sell all the items that he buys, how should he invest his money in order that may maximize his profit? Formulate a linear programming problem.

3. A dairy has its two plants one located at *P* and the other at *Q*. Each plant produces two types of products *A* and *B* in 1 kg packets. The capacity of two plants in number of packets per day are as follows:

Plants → Products ↓	P	Q
A	2000	1500
B	4000	6000

A market survey indicates that during the month of April, there will be a demand for 20000 packets of *A* and 16000 packets of *B*. The operating cost per day of running plants *P* and *Q* are respectively Rs.4000 and Rs.7500. How many days should the firm run each plant in the month of April so that the production cost is minimized while still meeting the market demand? Formulate a Linear programming problem.

4. A factory manufactures two articles *A* and *B*. To manufacture the article *A*, a certain machine has to be worked for 1 hour and 30 minutes and in addition a craftsman has to work for 2 hours. To manufacture the article *B*, the machine has to be worked for 2 hours and 30 minutes and in addition the craftsman has to work for 1 hour and 30 minutes. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article A is Rs.5 and that on each article *B* is Rs.4. If all the articles produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as a linear programming problem.

6.8 GEOMETRIC APPORACH OF LINEAR PROGRAMMING PROBLEM

Let us consider a simple problem in two variables *x* and *y*. Find *x* and *y* which satisfy the following equations

$$x + y = 4$$
$$3x + 4y = 14$$

Solving these equations, we get $x=2$ and $y=2$. What happens when the number of equations and variables are more?

Can we find a unique solution for such system of equations?

However, a unique solution for a set of simultaneous equations in n-variables can be obtained if there are exactly n-relations. What will happen when the number of relations is greater than or less then n ?

A unique solution will not exist, but a number of trial solutions can be found. Again, if the number of relations are greater than or less than the number of variables involved and the relation are in the form of inequalities.

Can we find a solution for such a system?

Whenever the analysis of a problem leads to minimising or maximising a linear expression in which the variable must obey a collection of linear inequalities, a solution may be obtained using linear programming techniques. One way to solve linear programming problems that involve only two variables is geometric approach called **graphical solution of the linear programming problem.**

6.9 SOLUTION OF LINEAR PROGRAMMING PROBLEMS

In the previous section we have seen the problems in which the number of relations are not equal to the number of variables and many of the relations are in the form of inequation (i.e., $\leq$ or $\geq$) to maximise (or minimise) a linear function of the variables subject to such conditions.

Now the question is how one can find a solution for such problems?

To answer this questions, let us consider the system of equations and inequations (or inequalities).

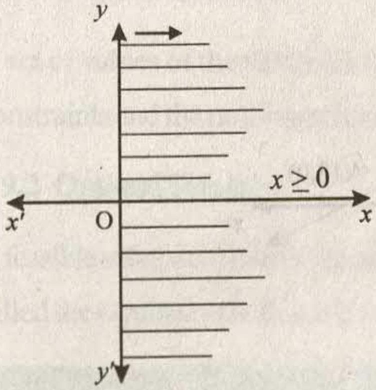


Fig. 6.11

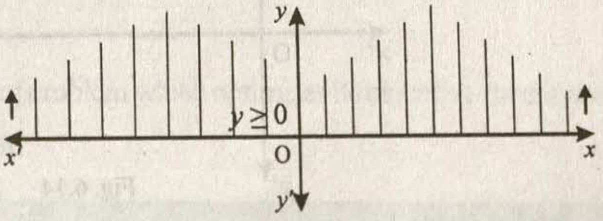


Fig. 6.12

We know that $x \geq 0$ represents a region lying towards the right of y - axis including the y - axis.

Similarly, the region represented by $y \geq 0$, lies above the x - axis including the x-axis.

The question arises: what region will be represented by $x \geq 0$ and $y \geq 0$ simultaneously.

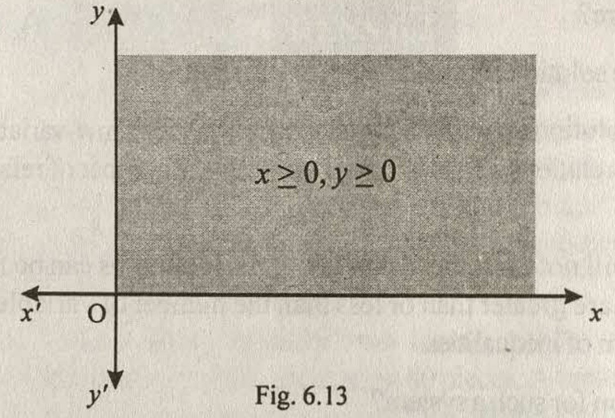


Fig. 6.13

Obviously, the region given by $x \geq 0, y \geq 0$ will consist of those points which are common to both $x \geq 0$ and $y \geq 0$. It is the first quadrant of the plane.

Next, we consider the graph of the equation $x + 2y \leq 8$. For this, first we draw the line $x + 2y = 8$ and then find the region satisfying $x + 2y \leq 8$.

Usually we choose x = 0 and calculate the corresponding value of y and choose y = 0 and calculate the corresponding value of x to obtain two sets of values (This method fails, if the line is parallel to either of the axes or passes through the origin. In that case, we choose any arbitrary value for x and choose y so as to satisfy the equation).

Plotting the points (0,4) and (8,0) and joining them by a straight line, we obtain the graph of the line as given in the Fig. 6.14 below.

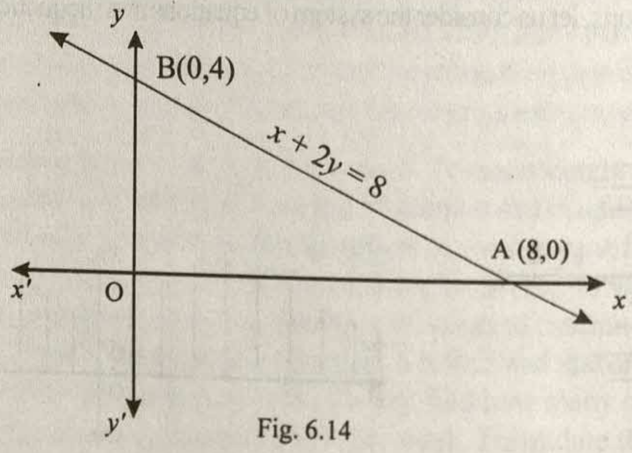


Fig. 6.14

We have already seen that $x \geq 0$ and $y \geq 0$ represents the first quadrant. The graph given by $x+2y<8$ lies towards that side of the line $x+2y=8$ in which the origin is situated because any point in this region will satisfy the inequality. Hence the shaded region in the Fig. 6.15 represents $x \geq 0, y \geq 0$ and $x + 2y \leq 8$ simultaneously.

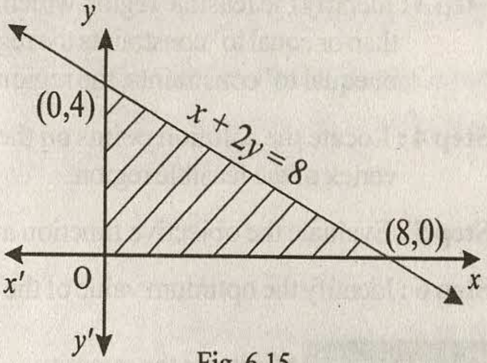


Fig. 6.15

Similarly, if we have to consider the regions bounded by $x \geq 0, y \geq 0$ and $x+2y \geq 8$, then it will lie in the first quadrant and on that side of the line $x+2y=8$ in which the origin is not located. The graph is shown by the shaded region, in Fig. 6.16

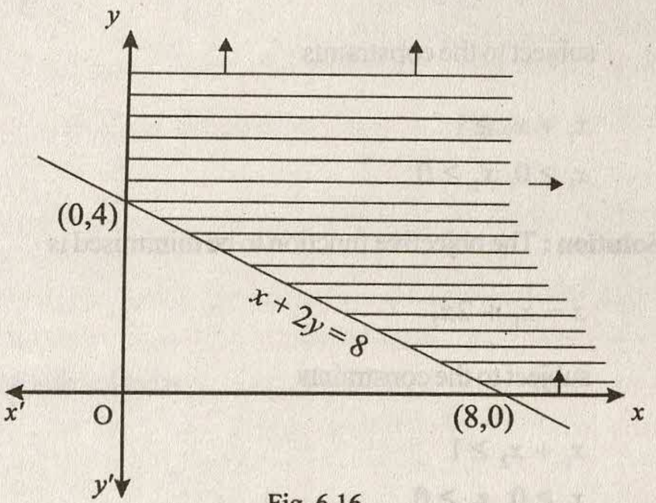


Fig. 6.16

The shaded region in which all the given constraints are satisfied is called the **feasible region.**

6.9.1 Feasible Solution

A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution** of the problem.

6.9.2 Optimal Solution

A feasible solution of a linear programming problem which optimises its objective functions is called the **optimal solution** of the problem.

*Note : If none of the feasible solutions maximise (or minimise) the objective function, or if there are no feasible solutions, then the linear programming problem has **no solution.***

In order to find a graphical solution of the linear programming problem, following steps be employed.

Step 1 : Formulate the linear programming problem.

Step 2 : Graph the constraints (inequalities), by the method discussed above.

Step 3 : Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to' constraints the region is generally below the lines and 'for greater than or equal to' constraints, the region is above the lines.

Step 4 : Locate the solution points on the feasible region. These points always occur at the vertex of the feasible region.

Step 5 : Evaluate the objective function at each of the vertex (corner point)

Step 6 : Identify the optimum value of the objective function.

Example 6.13 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities, which is as follows :

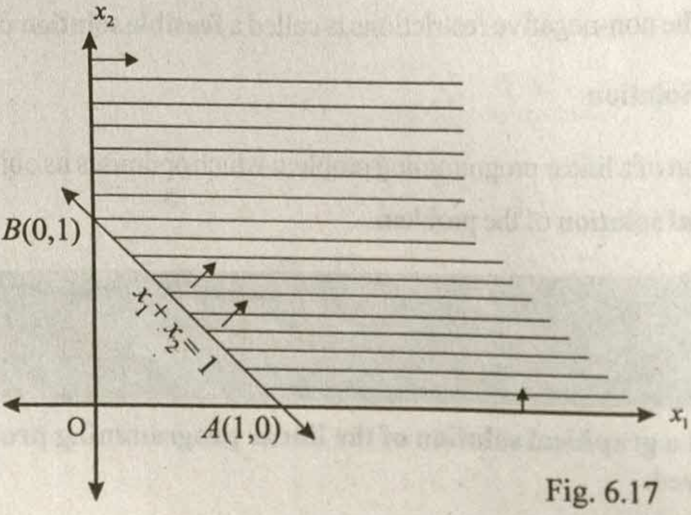


Fig. 6.17

As we have discussed earlier that the region satisfied by $x_1 \geq 0$ and $x_2 \geq 0$ is the first quadrant and the region satisfied by the line $x_1 + x_2 \geq 1$ along with $x_1 \geq 0$, $x_2 \geq 0$ will be on that side of the line $x_1 + x_2 = 1$ in which the origin is not located. Hence, the shaded region is our feasible solution because every point in this region satisfies all the constraints. Now, we have to find optimal solution. The vertex of the feasible region are *A* (1,0) and *B* (0,1).

The value of *z* at $A = 1$

The value of *z* at $B = 2$

Take any other point in the feasible region say (1,1), (2,0), (0,2) etc. We see that the value of *z* is minimum at *A* (1,0).

Example 6.14 Minimise the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function to be minimised is

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 3$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities (as discussed earlier) which is as follows :

The shaded region is the feasible region. Every point in the region satisfies all the mathematical inequalities and hence the feasible solution.

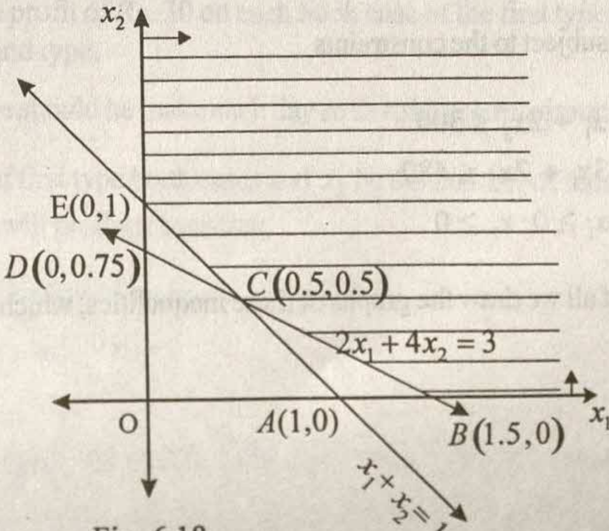


Fig. 6.18

Now, we have to find the optimal solution.

The value of z at $B\ (1.5, 0)$ is 1.5

The value of z at $C\ (0.5, 0.5)$ is 1.5

The value of z at E $(0,1)$ is 2

If we take any point on the line $2x_1 + 4x_2 = 3$ between B and C we will get $\frac{3}{2}$ and elsewhere in the feasible region greater than $\frac{3}{2}$. Of course, the reason any feasible point (between B and C) on $2x_1 + 4x_2 = 3$ minimizes the objective function (equation) $z = x_1 + 2x_2$ is that the two lines are parallel (both have slope $-\frac{1}{2}$). Thus this linear programming problem has infinitely many solutions and two of them occur at the vertices.

Example 6.15 Maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \leq 300$$
$$3x_1 + 2x_2 \leq 480$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : The objective function is to maximise

$$z = 0.25x_1 + 0.45x_2$$

subject to the constraints

$$x_1 + 2x_2 \leq 300$$
$$3x_1 + 2x_2 \leq 480$$
$$x_1 \geq 0,\ x_2 \geq 0$$

First of all we draw the graphs of these inequalities, which is as follows :

The shaded region $OABC$ is the feasible region. Every point in the region satisfies all the mathematical inequations and hence the feasible solutions.

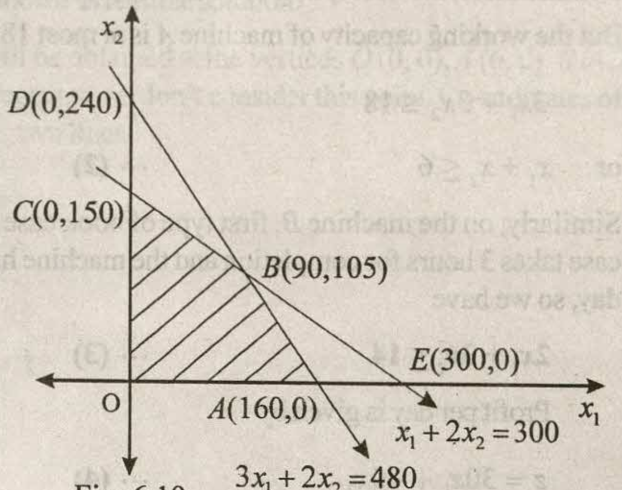


Fig. 6.19

Now, we have to find the optimal solution.

The value of z at A (160, 0) is 40.00

The value of z at B (90, 105) is 69.75.

The value of z at C (0, 150) is 67.50

The value of z at O (0, 0) is 0.

If we take any other value from the feasible region say (60, 120), (80, 80) etc. we see that still the maximum value is 69.75 obtained at the vertex B (90, 105) of the feasible region.

Note : For any linear programming problem that has a solution, the following general rule is true.

If a linear programming problem has a solution it is located at a vertex of the feasible region. If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the feasible region. In either case, the value of the objective function is unique.

Example 6.16 In a small scale industry a manufacturer produces two types of book cases. The first type of book case requires 3 hours on machine A and 2 hours on machines B for completion, whereas the second type of book case requires 3 hours on machine A and 3 hours on machine B. The machine A can run at the most for 18 hours while the machine B for at the most 14 hours per day. He earns a profit of Rs. 30 on each book case of the first type and Rs. 40 on each book case of the second type.

How many book cases of each type should he make each day so as to have a maximum porfit?

Solution : Let x_1 be the number of first type book cases and x_2 be the number of second type book cases that the manufacturer will produce each day.

Since x_1 and x_2 are the number of book cases so

$$x_1 \geq 0, x_2 \geq 0 \qquad \cdots \textbf{(1)}$$

Since the first type of book case requires 3 hours on machine A, therefore, x_1 book cases of first type will require $3x_1$ hours on machine A. second type of book case also requires 3 hours on machine A, therefore, x_2 book cases of second type will require $3x_2$ hours on machine A. But the working capacity of machine A is at most 18 hours per day, so we have

$$3x_1 + 3x_2 \le 18$$

or $\quad x_1 + x_2 \le 6 \qquad \ldots \textbf{(2)}$

Similarly, on the machine B, first type of book case takes 2 hours and second type of book case takes 3 hours for completion and the machine has the working capacity of 14 hours per day, so we have

$$\mathbf{2x_1 + 3x_2 \le 14} \qquad \ldots \textbf{(3)}$$

Profit per day is given by

$$z = 30x_1 + 40x_2 \qquad \ldots \textbf{(4)}$$

Now, we have to determine x_1 and x_2 such that

Maximize $z = 30x_1 + 40x_2$ (objective function) subject to the conditions

$$\left.\begin{array}{l} x_1 + x_2 \le 6 \\ 2x_1 + 3x_2 \le 14 \\ x_1 \ge 0,\ x_2 \ge 0 \end{array}\right\} \textbf{constraints}$$

We use the graphical method to find the solution of the problem. First of all we draw the graphs of these inequalities, which is as follows :

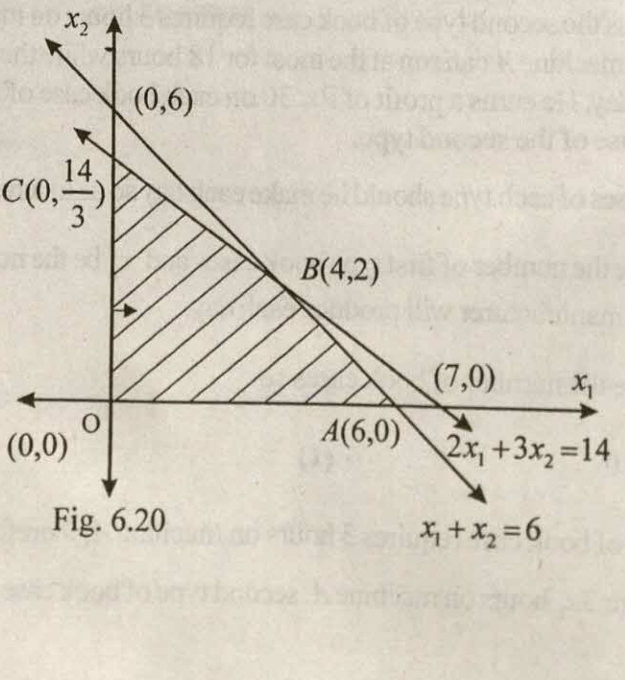


Fig. 6.20

The shaded region OABC is the feasible region. Every point in the region satisfies all the mathematical inequations and hence known as feasible solution.

We know that the optimal solution will be obtained at the vertices $O\,(0,0)$, $A\,(6,0)$. $B\,(4,2)$. Since the co-ordinates of C are not integers so we don't consider this point. Co-ordinates of B are calculated as the intersection of the two lines.

Now the profit at O is zero.

Profit at A $= 30 \times 6 + 40 \times 0$

$= 180$

Profit at B $= 30 \times 4 + 40 \times 2$

$= 120 + 80$

$= 200$

Thus the small scale manufacturer gains the maximum profit of Rs.200 if he prepares 4 first type book cases and 2 second type book cases.

Example 6.17 Solve Example 6.10 by graphical method.

Solution : From Example 6.10 we have

(minimise) $z = 6000\,x_1 + 4000\,x_2$ (objective function)

subject to the constraints

$$\left.\begin{aligned} 3x_1 + x_2 &\geq 24 \\ x_1 + x_2 &\geq 16 \\ x_1 + 3x_2 &\geq 24 \\ x_1 \geq 0,\ x_2 &\geq 0 \end{aligned}\right\} \text{(constraints)}$$

First we observe that any point $(x_{1,}\ x_2)$ lying in the first quadrant clearly satisfies the constraints

$x_1 \geq 0$, and $x_2 \geq 0$,

Now, we plot the bounding lines

$$3x_1 + x_2 = 24$$

or $$\frac{x_1}{8} + \frac{x_2}{24} = 1$$

Any point on or above the line $3x_1 + x_2 = 24$ satisfies the constraint $3x_1 + x_2 \geq 24$ (Fig. 6.21)

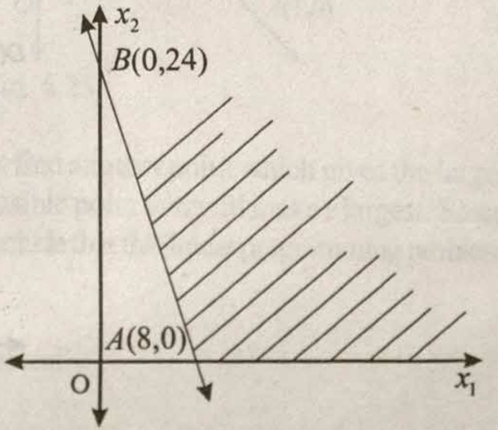


Fig. 6.21

Similarly any point on or above the line $x_1 + x_2 = 16$ satisfies the constraint $x_1 + x_2 \geq 16$ (See Fig. 6.22)

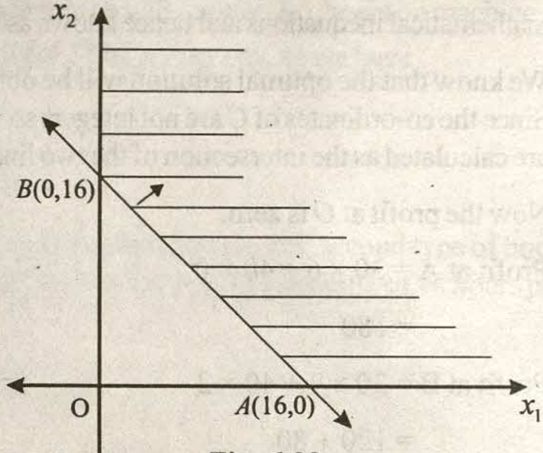


Fig. 6.22

Again any point on or above the line $x_1 + 3x_2 = 24$ satisfies the constraints $x_1 + 3x_2 \geq 24$ (See Fig. 6.23)

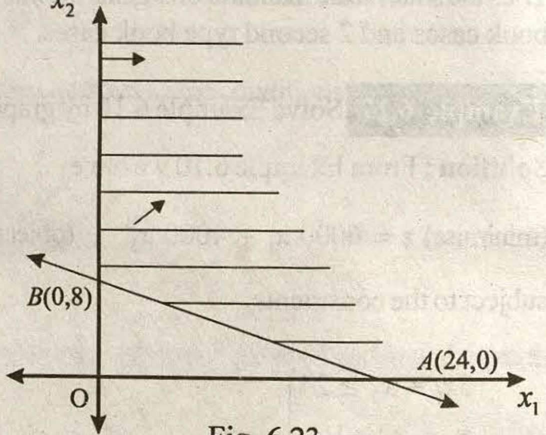


Fig. 6.23

Thus, combining all the above figures we get, the common shaded unbounded region. (See Fig. 6.24)

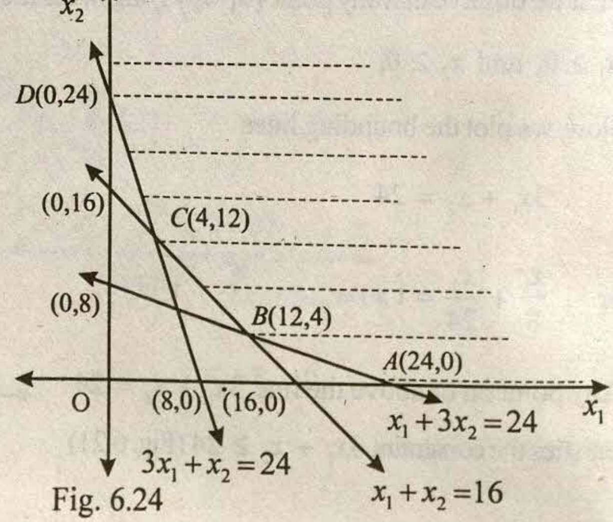


Fig. 6.24

Now the minimum value of

$$z = 6000x_1 + 4000x_2$$

is at one of the points A (24,0), B(12,4), C(4,12) and D(0,24)

At $A : z = 6000 \times 24 + 0 = 144{,}000$

At $B : z = 6000 \times 12 + 4000 \times 4 = 88{,}000$

At $C : z = 6000 \times 4 + 4000 \times 12 = 72{,}000$

At $D : z = 0 + 4000 \times 24 = 96{,}000$

Thus, we see that z is minimum at C (4, 12) where $x_1 = 4$ and $x_2 = 12$.

Hence, for minimum cost the firm should run plant P for 4 days and plant Q for 12 days. The minimum cost will be Rs.72,000.00

Example 6.18 Maximize the quantity

$$z = x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

Solutions : First we graph the constraints

$$x_1 + x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

The shaded portion is the set of feasible solution.

Now, we have to maximize the objective function.

The value of z at $A(1, 0)$ is 1.

The value of z at $B(0, 1)$ is 2.

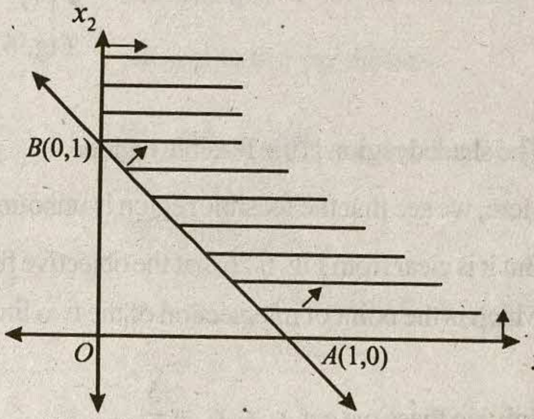


Fig. 6.25

If we take the value of z at any other point from the feasible region, say (1, 1) or (2, 3) or (5, 4) etc, then we notice that every time we can find another point which gives the larger value than the previous one. Hence, there is no feasible point that will make z largest. Since there is no feasible point that makes z largest, we conclude that this linear programming problem has no solution.

Example 6.19 Solve the following problem graphically.

Minimize $z = 2x_1 - 10x_2$

subject to the constraints

$$x_1 - x_2 \geq 0$$
$$x_1 - 5x_2 \leq -5$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution : First we graph the constraints

$$\begin{array}{lcl} x_1 - x_2 \geq 0 & & x_2 - x_1 \leq 0, \\ x_1 - 5x_2 \leq -5 & \textbf{or} & 5x_2 - x_1 \geq 5 \\ x_1 \geq 0,\ x_2 \geq 0 & & x_1 - x_2 = 0 \end{array}$$

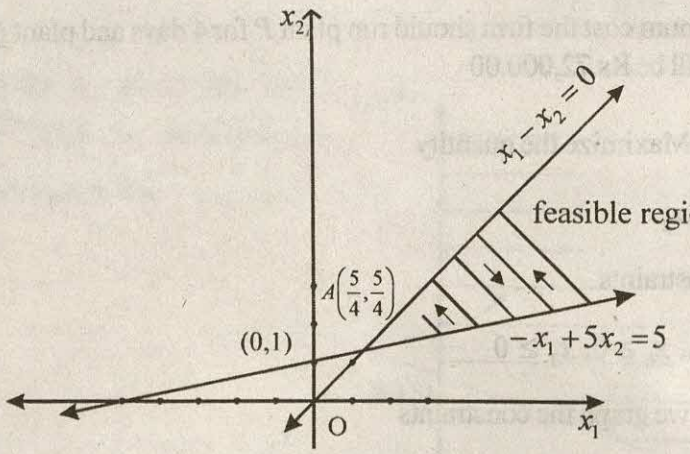


Fig. 6.26

The shaded region is the feasible region.

Here, we see that the feasible region is unbounded from one side.

But it is clear from Fig. 6.26 that the objective function attains its minimum value at the point A which is the point of intersection of the two lines $x_1 - x_2 = 0$ and $-x_1 + 5x_2 = 5$.

Solving these we get $x_1 = x_2 = \dfrac{5}{4}$

Hence, z is minimum when $x_1 = \dfrac{5}{4}$, $x_2 = \dfrac{5}{4}$, and its minimum value is

$$2 \times \frac{5}{4} - 10 \times \frac{5}{4} = -10.$$

Note : If we want to find max. z with these constraints then it is not possible in this case because the feasible region is unbounded from one side.

Notes

CHECK YOUR PROGRESS 6.4

Solve the following problems graphically

1. **Maximize** $z = 3x_1 + 4x_2$

   subject to the conditions

   $x_1 + x_2 \leq 40$
   $x_1 + 2x_2 \leq 60$
   $x_1 \geq 0,\ x_2 \geq 0$

2. **Maximize** $z = 2x_1 + 3x_2$

   subject to the conditions

   $x_1 + x_2 \leq 400$
   $2x_1 + x_2 \leq 600$
   $x_1 \geq 0, x_2 \geq 0$

3. **Minimize** $z = 60x_1 + 40x_2$

   subject to the conditions

   $3x_1 + x_2 \geq 24$
   $x_1 + x_2 \geq 16$
   $x_1 + 3x_2 \geq 24$
   $x_1 \geq 0,\ x_2 \geq 0$

4. **Maximize** $z = 20x_1 + 30x_2$

   subject to the conditions

   $x_1 + x_2 \leq 12,$
   $5x_1 + 2x_2 \leq 50$
   $x_1 + 3x_2 \leq 30,$
   $x_1 \geq 0,\ x_2 \geq 0$

5. **Maximize** $z = 50x_1 + 15x_2$

   subject to the conditions

   $5x_1 + x_2 \leq 100$
   $x_1 + x_2 \leq 60$
   $x_1 \geq 0, x_2 \geq 0$

6. **Minimize** $z = 4000x_1 + 7500x_2$

   subject to the conditions

   $4x_1 + 3x_2 \geq 40$
   $2x_1 + 3x_2 \geq 8$
   $x_1 \geq 0, x_2 \geq 0$

LET US SUM UP

A statement involving a sign of inequality like, $<, >, \leq, \geq$, is called an inequation.

- The equation $ax + by + c = 0$ is a straight line which divides the xy-plane into two half planes which are represented by $ax + by + c \geq 0$ and $ax + by + c \leq 0$
- By the term, solution of a system of linear inequations we mean, finding all values of the ordered pairs (x,y) for which each linear inequation of the system is satisfied.
- Linear programming is a technique followed by mathematicians to solve the optimisation problems.

- A set of values of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called a **feasible solution.**
- A feasible solution of a linear programming problem which optimises its objective function is called the **Optimal solution** of the problem.
- The optimal solution of a linear programming problem is located at a vertex of the set of feasible region.
- If a linear programming problem has multiple solutions, at least one of them is located at a vertex of the set of feasible region. But in all the cases the value of the objective function remains the same.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

Solve each of the following inequations graphically :

1. $x \geq -2$ 2. $y \leq 2.$ 3. $x < 3$ 4. $y \geq -3$

5. $5 - 3y \geq -4$ 6. $2x - 5 \leq 3.$ 7. $3x - 2y \leq 12$ 8. $\frac{x}{3} + \frac{y}{5} \geq 1.$

9. $2x - 3y \geq 0$ 10. $x + 2y \leq 0.$

Solve each of the following systems of linear inequations in two variables graphically.

11. $-1 \leq x \leq 3,\ 1 \leq y \leq 4.$

12. $2x + 3y \leq 6,\ 3x + 2y \leq 6.$

13. $6x + 5y \leq 150,$
$x + 4y \leq 80$
$x \leq 15,\ x \geq 0,\ y \geq 0.$

14. $3x + 2y \leq 24,\ x + 2y \leq 16$
$x + y \leq 10,\ x \geq 0,\ y \geq 0$

15. $x + y \geq 3,\ 7x + 6y \leq 42$
$x \leq 5,\quad y \leq 4$
$x \geq 0,\quad y \geq 0$

16. A dealer has Rs. 1500 only for a purchase of rice and wheat. A bag of rice costs Rs. 150 and a bag of wheat costs Rs. 120. He has a storage capacity of ten bags only and the dealer gets a profit of Rs.11 and Rs. 8 per bag of rice and wheat respectively. Formulate the problem as a linear programming problem to get the maximum profit.

17. A business man has Rs. 6000 at his disposal and wants to purchase cows and buffaloes to take up a business. The cost price of a cow is Rs. 900 and that of a buffalo is Rs.1200. The man can store fodder for the live stock to the extent of 40 quintals per week. A cow gives 10 litres of milk and buffalo gives 20 litres of milk per day. Profit per litre of milk of cow is 50 paise and per litre of the milk of a buffalo is 90 paise. If the consumption of fodder per cow is 1 quintal and per buffalo is 2 quintals a week, formulate the problem as a linear programming problem to find the number of live stock of each kind the man has to purchase so as to get maximum profit (assuming that he can sell all the quantity of milk, he gets from the livestock)

18. A factory manufactures two types of soaps each with the help of two machines *A* and *B*. *A* is operated for two minutes and *B* for 3 minutes to manufacture the first type, while the second type is manufactured by operating A for 3 minutes and B for 5 minutes. Each machine can be used for at most 8 hours on any day. The two types of soaps are sold at a profit of 25 paise and 50 paise each respectively. How many soaps of each type should the factory produce in a day so as to maximize the profit (assuming that the manufacturer can sell all the soaps he can manufacture). Formulate the problem as a linear programming problem.

19. Determine two non-negative rational numbers such that their sum is maximum provided that their difference exceeds four and sum of three times the first number and the second should be less than or equal to 9. Formulate the problem as a linear programming problem.

20. Vitamins *A* and *B* are found in two different foods *E* and *F*. One unit of food *E* contains 2 units of vitamin *A* and 3 units of vitamin *B*. One unit of food *F* contains 4 units of vitamin *A* and 2 units of vitamin *B*. One unit of food *E* and *F* costs Rs.5 and Rs.2.50 respectively. The minimum daily requirements for a person of vitamin *A* and *B* is 40 units and 50 units respectively. Assuming that anything in excess of daily minimum requirement of vitamin *A* and *B* is not harmful, find out the optimal mixture of food *E* and *F* at the minimum cost which meets the daily minimum requirement of vitamin *A* and *B*. Formulate this as a linear programming problem.

21. A dealer has Rs.1500 only for the purchase of rice and wheat. A bag of rice costs Rs. 180 and a bag of wheat costs Rs. 120. He has a storage capacity of 10 bags only and the dealer gets a profit of Rs.11 and Rs. 8 per bag of rice and wheat respectively. How he should spend his money in order to get maximum profit?

22. A machine producing either product *A* or *B* can produce *A* by using 2 units of chemicals and 1 unit of a compound and can produce *B* by using 1 unit of chemicals and 2 units of the compound. Only 800 units of chemicals and 1000 units of the compound are available. The profits available per unit of *A* and *B* are respectively Rs. 30 and Rs.20. Find the optimum allocation of units between *A* and *B* to maximise the total profit. Find the maximum profit.

23. Solve the following Linear programming problem graphically.

(a) Maximize $z = 25x_1 + 20x_2$

subject to the constraints

$$3x_1 + 6x_2 \le 50$$
$$x_1 + 2x_2 \le 10$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(b) Maximize $z = 9x_1 + 10x_2$

subject to the constraints

$$11x_1 + 9x_2 \le 9900$$
$$7x_1 + 12x_2 \le 8400$$
$$3x_1 + 8x_2 \le 4800$$
$$x_1 \ge 0,\ x_2 \ge 0$$

(c) Maximise $z = 22x_1 + 18x_2$

subject to the constraints

$$x_1 + x_2 \le 20,$$
$$3x_1 + 2x_2 \le 48$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Notes

ANSWERS

CHECK YOUR PROGRESS 6.1

1.

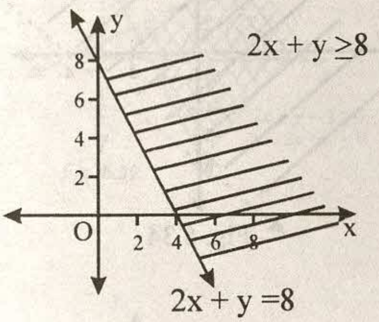


Fig 6.27

2.

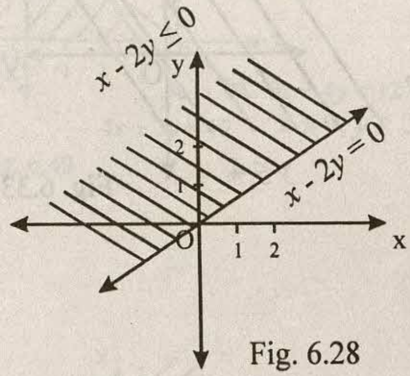


Fig. 6.28

3.

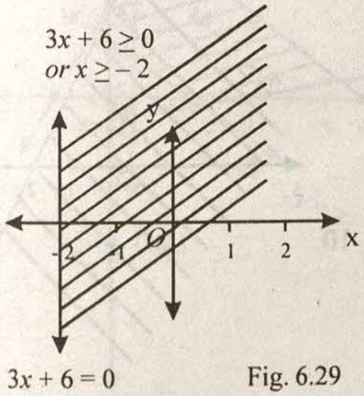


Fig. 6.29

4.

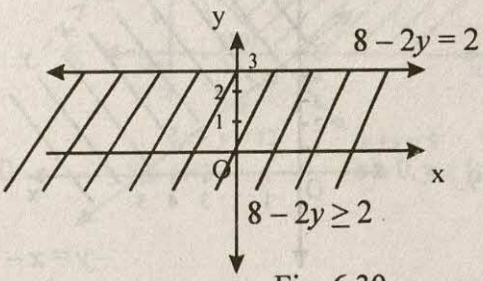


Fig. 6.30

5.

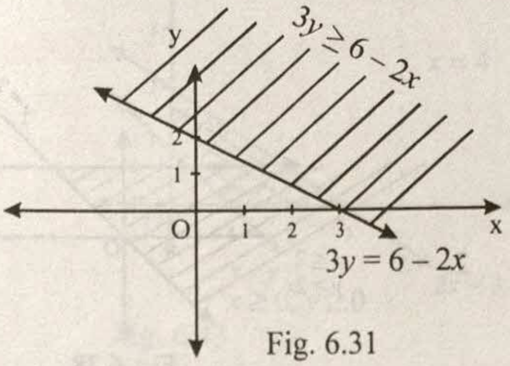


Fig. 6.31

6.

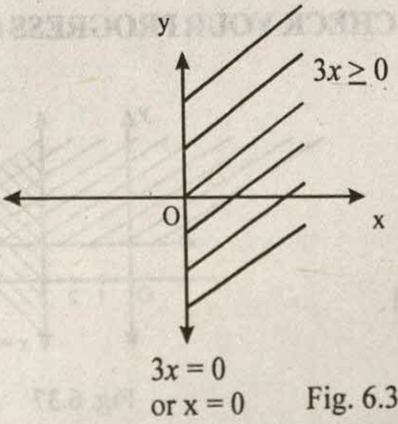


Fig. 6.32

7.

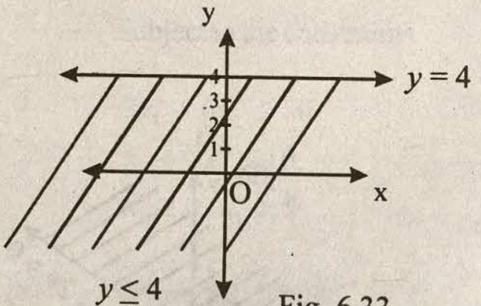


Fig. 6.33

8.

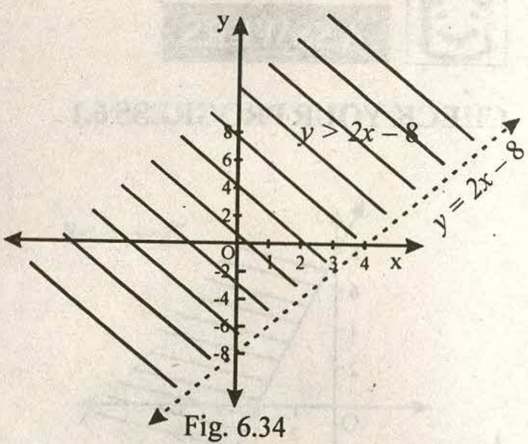


Fig. 6.34

9.

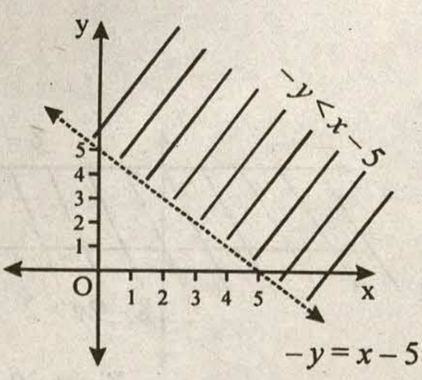


Fig. 6.35

10.

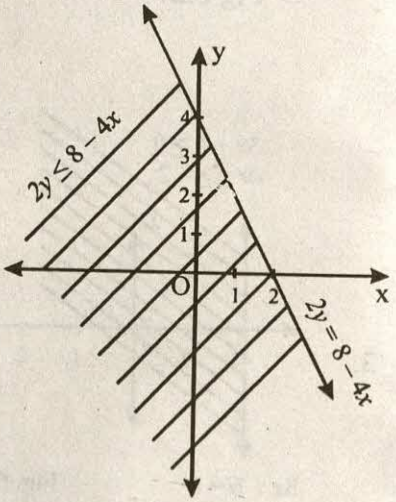


Fig. 6.36

CHECK YOUR PROGRESS 6.2

1.

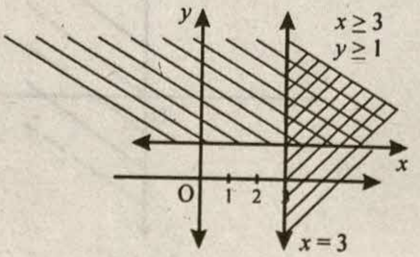


Fig. 6.37

2.

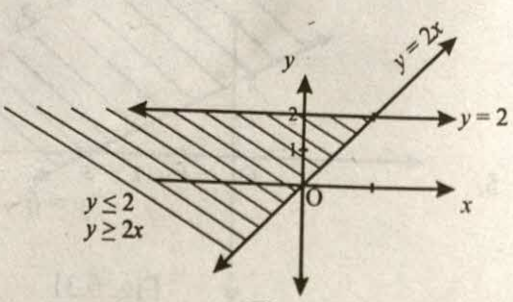


Fig. 6.38

(3)

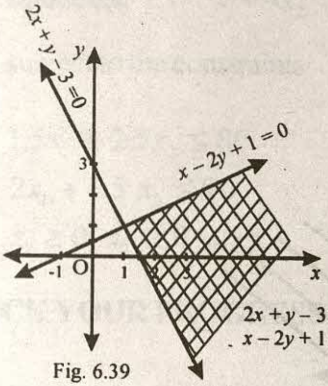


Fig. 6.39

(4)

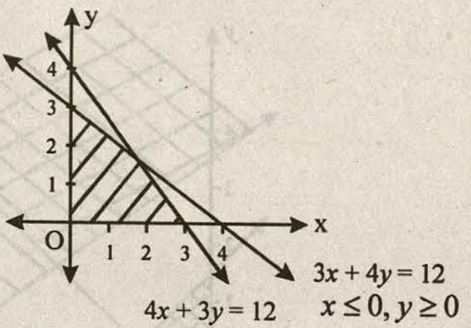


Fig. 6.40

5.

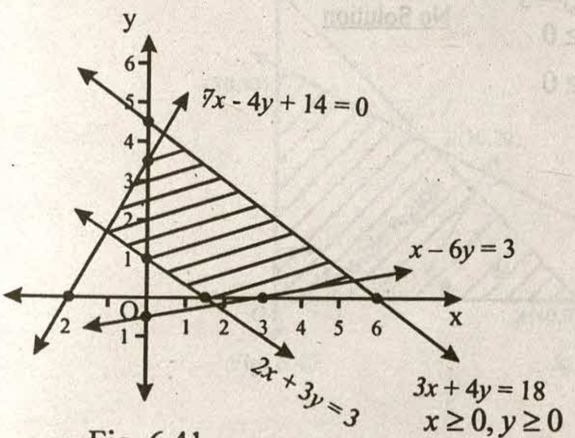


Fig. 6.41

6.

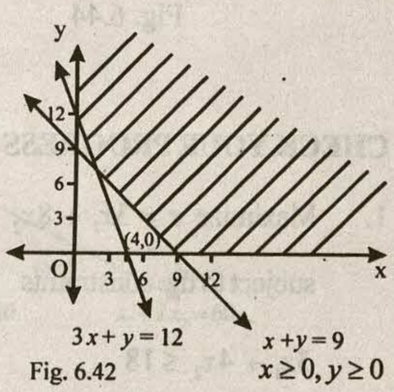


Fig. 6.42

7.

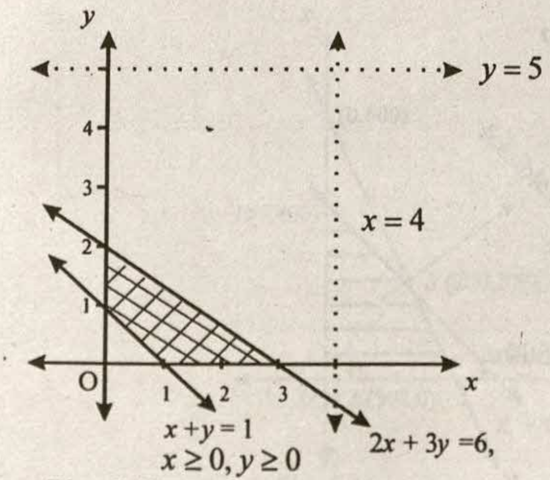


Fig. 6.43

8.

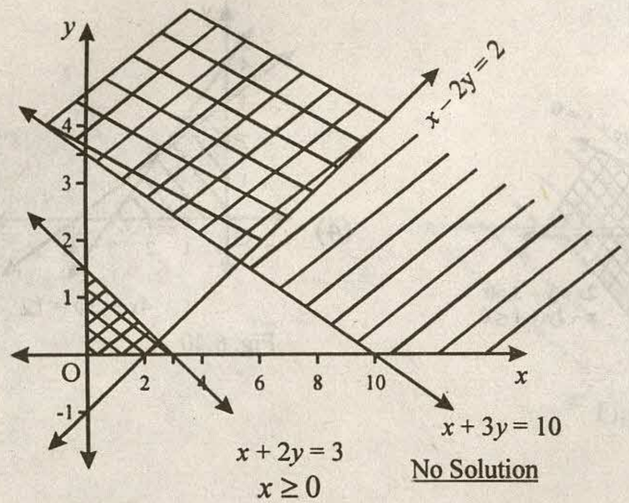


Fig. 6.44

CHECK YOUR PROGRESS 6.3

1. Maximize $z = 3x_1 + 8x_2$

   subject to the constraints

   $3x_1 + 4x_2 \le 18$
   $4x_1 + 5x_2 \le 21$
   $x_1 \ge 0,\ x_2 \ge 0.$

2. Maximize $z = 50x_1 + 15x_2$

   subject to the constraints

   $5x_1 + x_2 \le 100$
   $x_1 + x_2 \le 60$
   $x_1 \ge 0,\ x_2 \ge 0.$

3. Minimize $z = 4000x_1 + 7500x_2$

   subject to the constraints

   $4x_1 + 3x_2 \ge 40$
   $2x_1 + 3x_2 \ge 8$
   $x_1 \ge 0,\ x_2 \ge 0$

4. Maximize $z = 5x_1 + 4x_2$

subject to the constraints

$1.5x_1 + 2.5x_2 \le 80$
$2x_1 + 1.5x_2 \le 70$
$x_1 \ge 0, x_2 \ge 0$

CHECK YOUR PROGRESS 6.4

1.

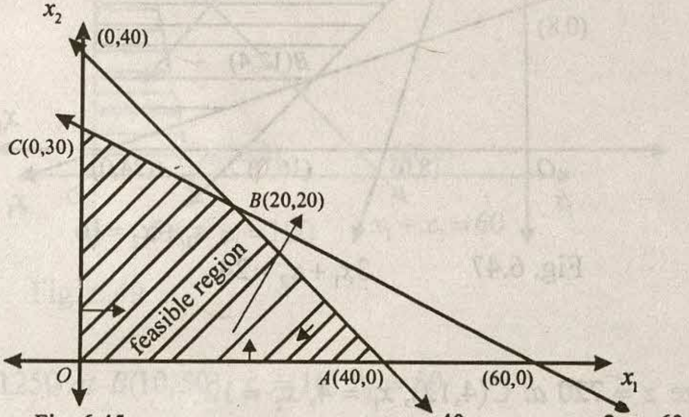


Fig. 6.45

Maximum $z = 140$ at $B(20,20)$

2.

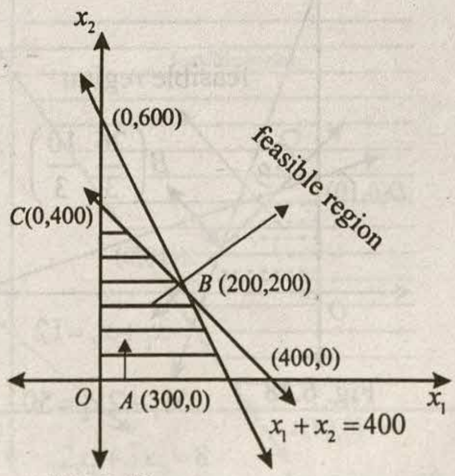


Fig. 6.46

Maximize $z = 1200$ at $C(0,400)$

3.

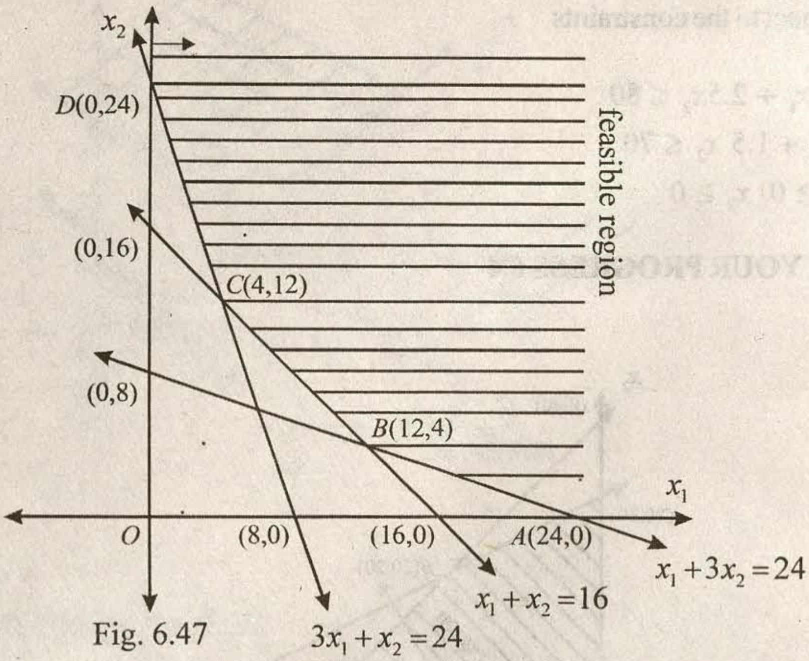


Fig. 6.47

Minimize $z = 720$ *at* $C(4,12)$, $x_1 = 4$, $x_2 = 12$

4.

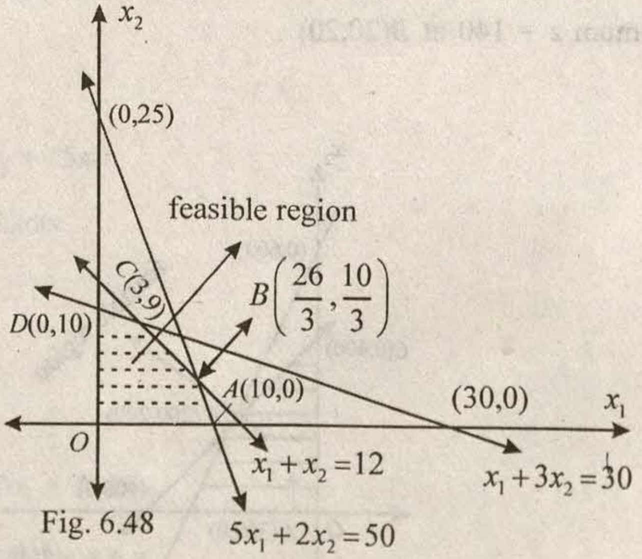


Fig. 6.48

Maximum $z = 330$ *at* $C(3,9)$, $x_1 = 3$, $x_2 = 9$

5.

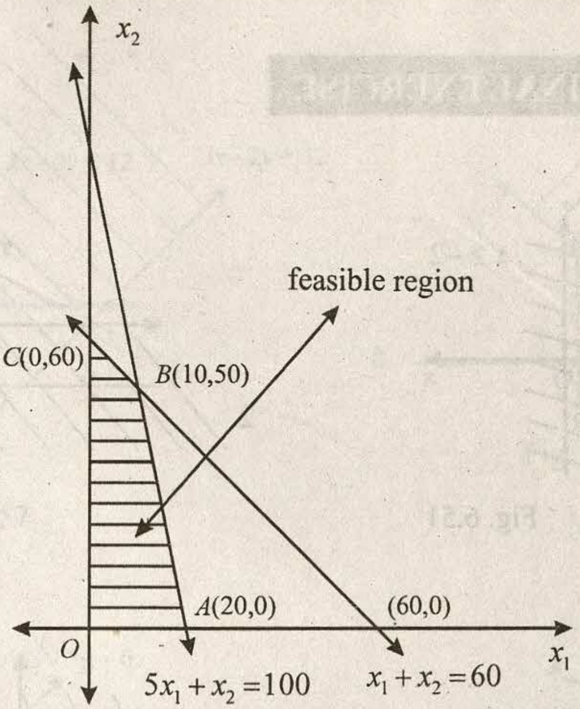


Fig. 6.49

Maximum $z = 1250$ *at* $B(10,50)$, $x_1 = 10$, $x_2 = 50$

6.

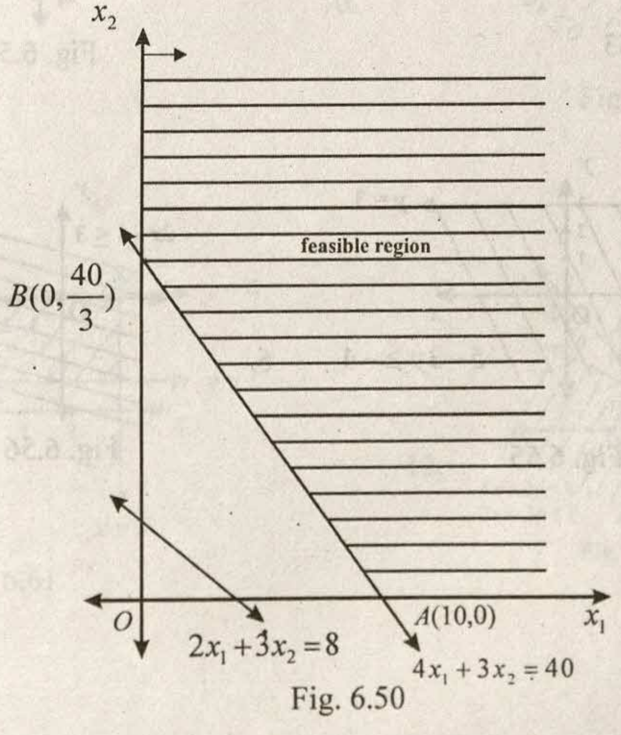


Fig. 6.50

Maximum z = 40,000 at A(10,0), $x_1 = 10$, $x_2 = 0$

TERMINAL EXERCISE

1.

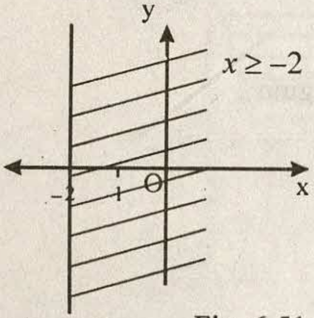


Fig. 6.51

2.

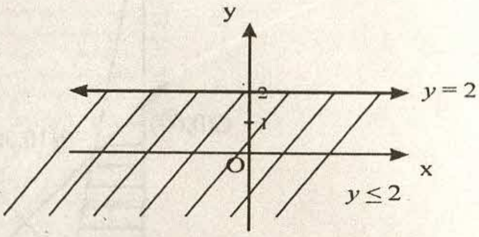


Fig. 6.52

3.

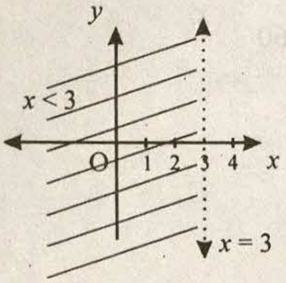


Fig. 6.53

4.

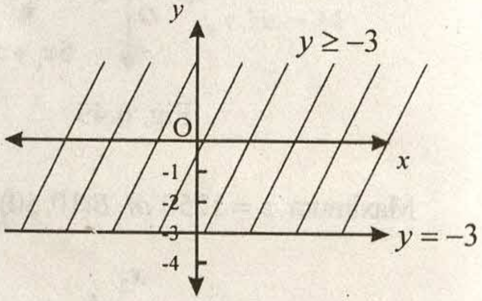


Fig. 6.54

5.

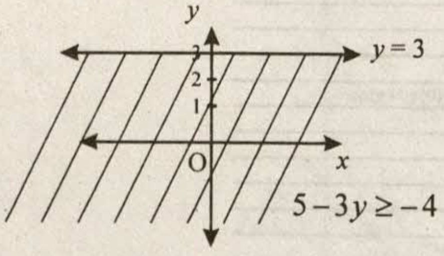


Fig. 6.55

6.

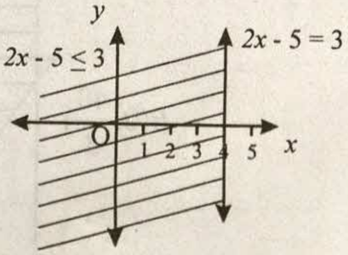


Fig. 6.56

7.

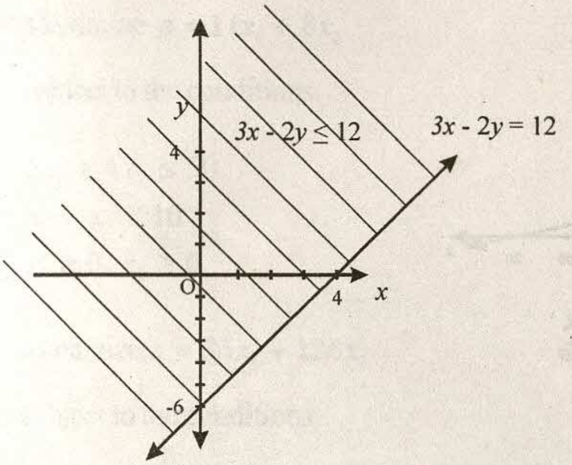


Fig. 6.57

8.

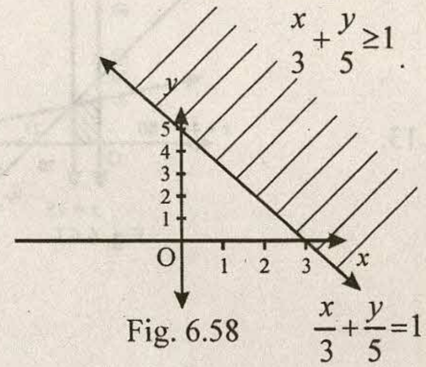


Fig. 6.58

9.

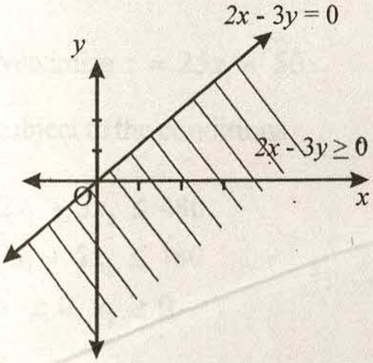


Fig. 6.59

10.

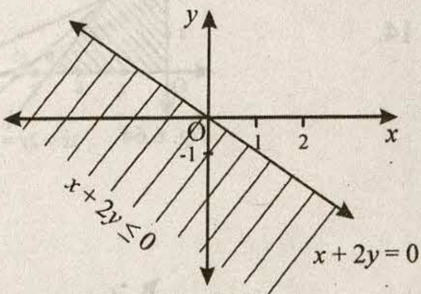


Fig. 6.60

11.

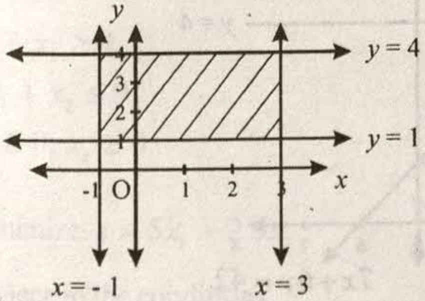


Fig. 6.61

12.

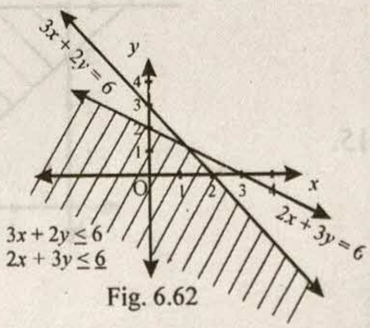


Fig. 6.62

13.

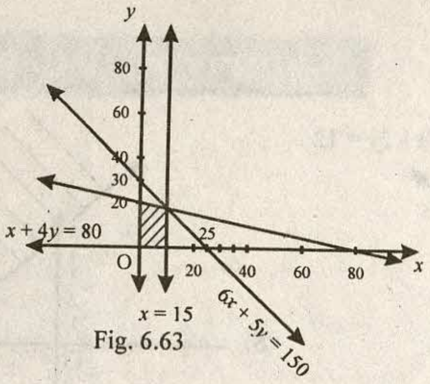


Fig. 6.63

14.

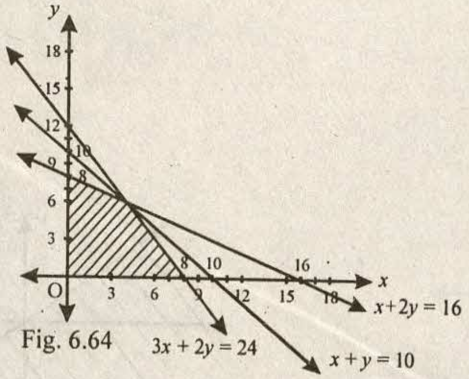


Fig. 6.64

15.

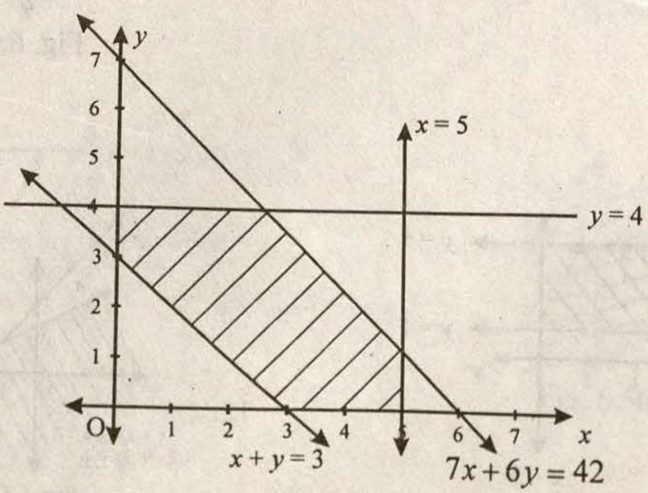


Fig. 6.65

16. Maximize $z = 11x_1 + 8x_2$

subject to the conditions

$5x_1 + 4x_2 \leq 50$
$x_1 + x_2 \leq 10$
$x_1 \geq 0,\ x_2 \geq 0$

17. Maximize $z = 35x_1 + 126x_2$

subject to the conditions

$3x_1 + 4x_2 \leq 20$
$7x_1 + 14x_2 \leq 40$
$x_1 \geq 0,\ x_2 \geq 0$

18. Maximize $z = 25x_1 + 50x_2$

subject to the conditions

$2x_1 + 3x_2 \leq 480$
$3x_1 + 5x_2 \leq 480$
$x_1 \geq 0,\ x_2 \geq 0$

19. Maximize $z = x_1 + x_2$

subject to the conditions

$x_1 - x_2 \geq 4$
$3x_1 + x_2 \leq 9$
$x_1 \geq 0,\ x_2 \geq 0$

20. Minimize $z = 5x_1 + 2.5x_2$

subject to the conditions

$x_1 + 2x_2 \geq 20$
$3x_1 + 2x_2 \geq 50$
$x_1 \geq 0,\ x_2 \geq 0$

Notes

21. Let x_1 bags of rice and x_2 bags of wheat to be purchased. We have,

Maximize $z = 11x_1 + 8x_2$

Subject to the constraints

$3x_1 + 2x_2 \leq 25$
$x_1 + x_2 \leq 10$
$x_1 \geq 0, x_2 \geq 0$

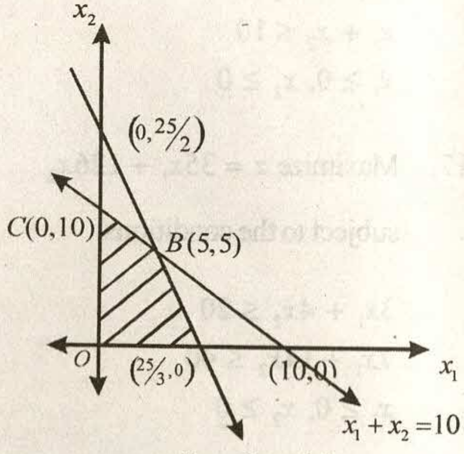


Fig. 6.66

Maximum $z = 95$ *at* $B(5,5)$, $x_1 = x_2 = 5$

22. Maximize $z = 30x_1 + 20x_2$
subject to the constraints

$2x_1 + x_2 \leq 800$
$x_1 + 2x_2 \leq 1000$
$x_1 \geq 0, x_2 \geq 0$

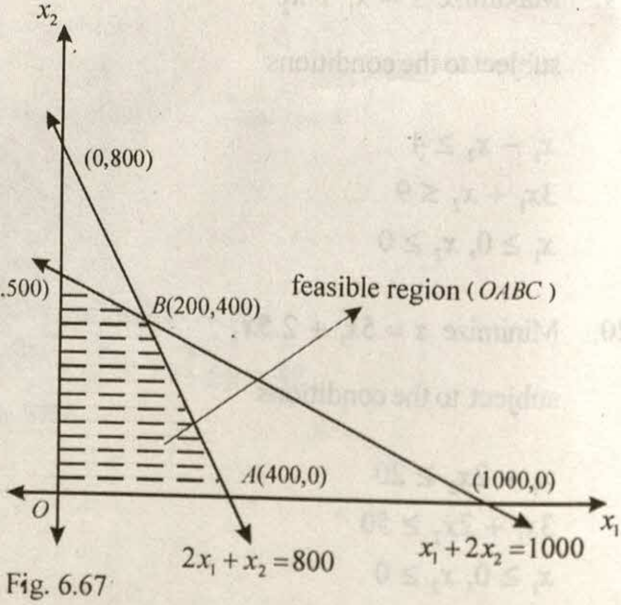


Fig. 6.67

Maximum z = 14000 at B (200,400). $x_1 = 200, x_2 = 400$

23. (a)

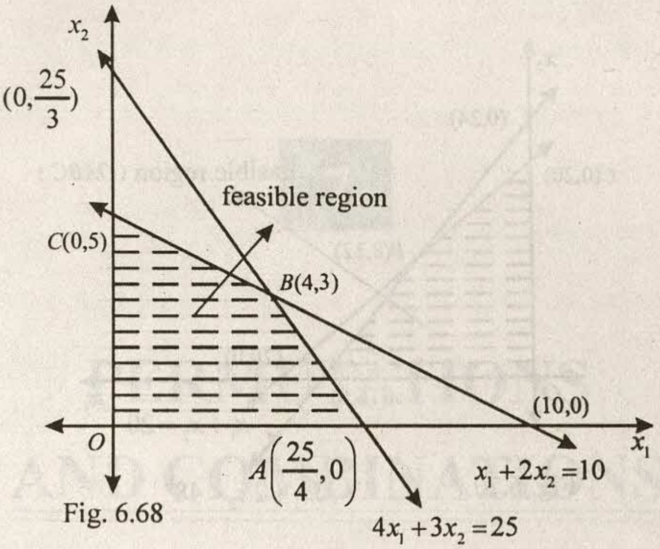


Fig. 6.68

Maximum $z = 160$ *at* $B(4,3)$, $x_1 = 4$ $x_2 = 3$

(b)

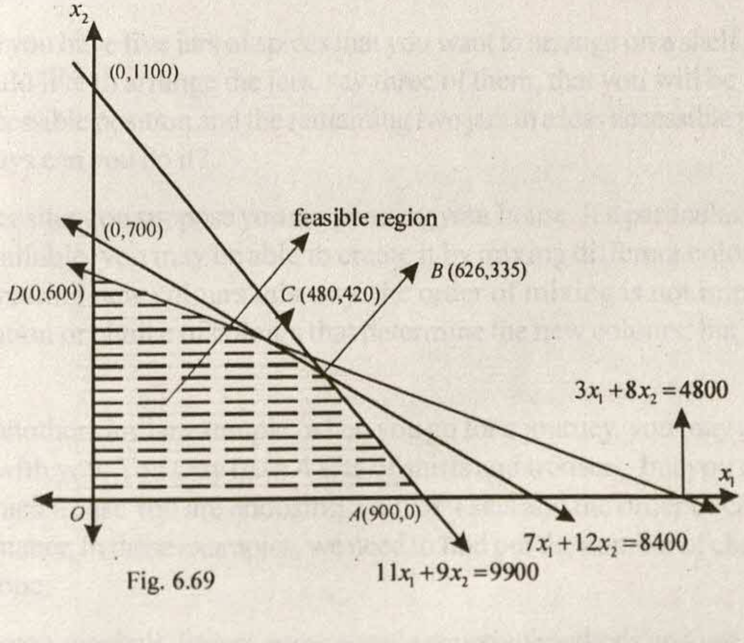


Fig. 6.69

A (900,0) D(0,600) B (626, 335), O(0, 0) and C (480, 420)

Maximum $z = 8984$ *at* $B(626,335)$ $x_1 = 626$, $x_2 = 335$

(c)

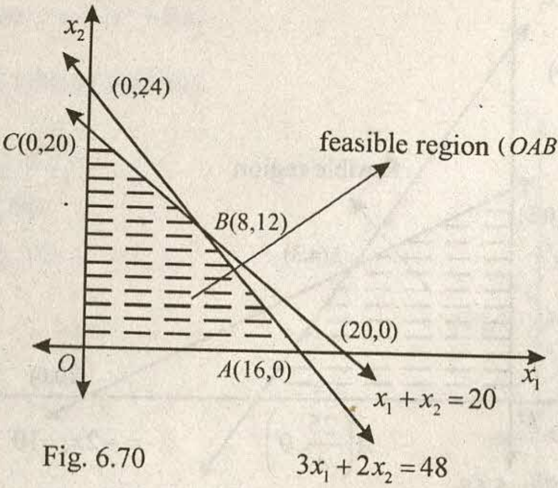


Fig. 6.70

Maximum $z = 392$ *at* $B(8, 12)$ $x_1 = 8$, $x_2 = 12$

7

PERMUTATIONS AND COMBINATIONS

One day, I wanted to travel from Bangalore to Allahabad by train. There is no direct train from Bangalore to Allahabad, but there are trains from Bangalore to Itarsi and from Itarsi to Allahabad. From the railway timetable I found that there are two trains from Bangalore to Itarsi and three trains from Itarsi to Allahabad. Now, in how many ways can I travel from Bangalore to Allahabad?

This is a **counting problem** which come under the branch of Mathematics called **combinatorics.**

Suppose you have five jars of spices that you want to arrange on a shelf in your kitchen. You would like to arrange the jars, say three of them, that you will be using often in a more accessible position and the remaining two jars in a less accessible position. In how many ways can you do it?

In another situation suppose you are painting your house. If a particular shade or colour is not available, you may be able to create it by mixing different colours and shades. While creating new colours this way, the order of mixing is not important. It is the combination or choice of colours that determine the new colours; but not the order of mixing.

To give another similar example, when you go for a journey, you may not take all your dresses with you. You may have 4 sets of shirts and trousers, but you may take only 2 sets. In such a case you are choosing 2 out of 4 sets and the order of choosing the sets doesn't matter. In these examples, we need to find out the number of choices in which it can be done.

In this lesson, we shall discuss some simple counting methods and use them in solving such simple counting problems.

OBJECTIVES

After studying this lesson, you will be able to :

- find out the number of ways in which a given number of objects can be arranged;
- state the Fundamental Principle of Counting;
- define n! and evaluate it for defferent values of n;
- state that permutation is an arrangement and write the meaning of ^nP_r;
- state that $^nP_r = \dfrac{n!}{(n-r)!}$ and apply this to solve problems;
- show that $(i)\ (n+1)\,^nP_n = {}^{n+1}P_n \quad (ii)\ ^nP_{r+1} = (n-r)\,^nP_r$;
- state that a combination is a selection and write the meaning of ^nC_r;
- distinguish between permutations and combinations;
- derive $^nC_r = \dfrac{n!}{r!(n-r)!}$ and apply the result to solve problems;
- derive the relation $^nP_r = r!\,^nC_r$;
- verify that $^nC_r = {}^nC_{n-r}$ and give its interpretation; and
- derive $^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$ and apply the result to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Number Systems
- Four Fundamental Operations

7.1 COUNTING PRINCIPLE

Let us now solve the problem mentioned in the introduction. We will write t_1, t_2 to denote trains from Bangalore to Itarsi and T_1, T_2, T_3, for the trains from Itarsi to Allahabad. Suppose I take t_1 to travel from Bangalore to Itarsi. Then from Itarsi I can take T_1 or T_2 or T_3. So the possibilities are t_1T_1, t_2T_2 and t_3T_3 where t_1T_1 denotes travel from Bangalore to Itarsi by t_1 and travel from Itarsi to Allahabad by T_1. Similarly, if I take t_2 to travel from Bangalore to Itarsi, then the possibilities are t_2T_1, t_2T_2 and t_2T_3. Thus, in all there are $6\,(2\times 3)$ possible ways of travelling from Bangalore to Allahabad.

Here we had a small number of trains and thus could list all possibilities. Had there been 10 trains from Bangalore to Itarsi and 15 trains from Itarsi to Allahabad, the task would have been very tedious. Here the **Fundamental Principle of Counting** or simply the **Counting Principle** comes in use :

If any event can occur in *m* ways and after it happens in any one of these ways, a second event can occur in *n* ways, then both the events together can occur in $m \times n$ ways.

Example 7.1 How many multiples of 5 are there from 10 to 95 ?

Solution : As you know, multiples of 5 are integers having 0 or 5 in the digit to the extreme right (i.e. the unit's place).

The first digit from the right can be chosen in 2 ways.

The second digit can be any one of 1,2,3,4,5,6,7,8,9.

i.e. There are 9 choices for the second digit.

Thus, there are $2 \times 9 = 18$ multiples of 5 from 10 to 95.

Example 7.2 In a city, the bus route numbers consist of a natural number less than 100, followed by one of the letters A, B, C, D, E and F. How many different bus routes are possible?

Solution : The number can be any one of the natural numbers from 1 to 99.
There are 99 choices for the number.

The letter can be chosen in 6 ways.

$\therefore$ Number of possible bus routes are $99 \times 6 = 594$.

CHECK YOUR PROGRESS 7.1

1. (a) How many 3 digit numbers are multiples of 5?

   (b) A coin is tossed thrice. How many possible outcomes are there?

   (c) If you have 3 shirts and 4 pairs of trousers and any shirt can be worn with any pair of trousers, in how many ways can you wear your shirts and pairs of trousers?

   (d) A tourist wants to go to another country by ship and return by air. She has a choice of 5 different ships to go by and 4 airlines to return by. In how many ways can she perform the journey?

2. (a) In how many ways can two vacancies be filled from among 4 men and 12 women if one vacancy is filled by a man and the other by a woman?

   (b) Flooring and painting of the walls of a room needs to be done. The flooring can be done in 3 colours and painting of walls can be done in 12 colours. If any colour combination is allowed, find the number of ways of flooring and painting the walls of the room.

So far, we have applied the counting principle for two events. But it can be extended to three or more, as you can see from the following examples :

Example 7.3 There are 3 questions in a question paper. If the questions have 4,3 and 2 respectively, find the total number of solutions.

Solution : Here question 1 has 4 solutions,

question 2 has 3 solutions

and question 3 has 2 solutions.

$\therefore$ By the counting principle,

$$\text{total number of solutions} = 4\times3\times2$$
$$= 24$$

Example 7.4 Consider the word ROTOR. Whichever way you read it, from left to right or from right to left, you get the same word. Such a word is known as *palindrome*. Find the maximum possible number of 5-letter palindromes.

Solution : The first letter from the right can be chosen in 26 ways because there are 26 alphabets.

Having chosen this, the second letter can be chosen in 26 ways

$\therefore$ The first two letters can chosen in $26\times26=676$ ways

Having chosen the first two letters, the third letter can be chosen in 26 ways.

$\therefore$ All the three letters can be chosen in $676\times26=17576$ ways.

It implies that the maximum possible number of five letter palindromes is 17576 because the fourth letter is the same as the second letter and the fifth letter is the same as the first letter.

Note : In Example 7.4 we found the maximum possible number of five letter palindromes. There cannot be more than 17576. But this does not mean that there are 17576 palindromes. Because some of the choices like CCCCC may not be meaningful words in the English language.

Example 7.5 How many 3-digit numbers can be formed with the digits 1,4,7,8 and 9 if the digits are not repeated.

Solution : Three digit number will have unit's, ten's and hundred's place.

Out of 5 given digits any one can take the unit's place.

This can be done in 5 ways. ... (*i*)

After filling the unit's place, any of the four remaining digits can take the ten's place.

This can be done in 4 ways. ... (*ii*)

After filling in ten's place, hundred's place can be filled from any of the three remaining digits.

This can be done in 3 ways. ... *(iii)*

∴ By counting principle, the number of 3 digit numbers = $5 \times 4 \times 3 = 60$

Let us now state the **General Counting Principle**

If there are *n* events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occured, the third event can occur in m_3 ways after the second event has ocurred, and so on, then all the *n* events can occur in $m_1 \times m_2 \times \ldots \times m_{n-1} \times m_n$ ways.

Example 7.6 Suppose you can travel from a place *A* to a place *B* by 3 buses, from place *B* to place *C* by 4 buses, from place *C* to place *D* by 2 buses and from place *D* to place *E* by 3 buses. In how many ways can you travel from *A* to *E*?

Solution : The bus from *A* to *B* can be selected in 3 ways.

The bus from *B* to *C* can be selected in 4 ways.

The bus from *C* to *D* can be selected in 2 ways.

The bus from *D* to *E* can be selected in 3 ways.

So, by the General Counting Principle, one can travel from *A* to *E* in $3 \times 4 \times 2 \times 3$ ways = 72 ways.

CHECK YOUR PROGRESS 7.2

1. (a) What is the maximum number of 6-letter palindromes?

   (b) What is the number of 6-digit palindromic numbers which do not have 0 in the first digit?

2. (a) In a school there are 5 English teachers, 7 Hindi teachers and 3 French teachers. A three member committee is to be formed with one teacher representing each language. In how many ways can this be done?

   (b) In a college students union election, 4 students are contesting for the post of President. 5 students are contesting for the post of Vice-president and 3 students are contesting for the post of Secretary. Find the number of possible results.

3. (a) How many three digit numbers greater than 600 can be formed using the digits 1,2,5,6,8 without repeating the digits?

   (b) A person wants to make a time table for 4 periods. He has to fix one period each for English, Mathematics, Economics and Commerce. How many different time tables can he make?

7.2 PERMUTATIONS

Suppose you want to arrange your books on a shelf. If you have only one book, there is only one way of arranging it. Suppose you have two books, one of History and one of Geography.

You can arrange the Geography and History books in two ways. Geography book first and the History book next, *GH* or History book first and Geography book next; *HG*. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say *GH*, you can put Mathematics book in one of the following ways: *MGH*, *GMH* or *GHM*. Similarly, corresponding to *HG*, you have three other ways of arranging the books. So, by the Counting Principle, you can arrange Mathematics, Geography and History books in 3×2 ways $= 6$ ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books.

In general, if you want to find the number of permutations of *n* objects $n\geq 1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3\times 2\times 1$ permutations of 3 objects. It may be that, there are $n\times(n-1)\times(n-2)\times\ldots\times 2\times 1$ permutations of *n* objects. In fact, it is so, as you will see when we prove the following result.

Theorem 7.1 The total number of permutations of *n* objects is $n\,(n-1)\ldots.2.1$.

Proof : We have to find the number of possible arrangements of *n* different objects.

The first place in an arrangement can be filled in *n* different ways. Once it has been done, the second place can be filled by any of the remaining (*n*–1) objects and so this can be done in (*n*–1) ways. Similarly, once the first two places have been filled, the third can be filled in (*n*–2) ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object.

Using the counting principle, the total number of arrangements of *n* different objects is $n(n-1)(n-2)\ldots\ldots..\,2.1.$ .....(7.1)

The product $n\,(n-1)\ldots 2.1$ occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by ***n*!** (or by $\lfloor n$) read as ***n* factorial.**

$$n! = n\,(n-1)\ldots 3.2.1$$

Here is an example to help you familiarise yourself with this notation.

Example 7.7 Evaluate (a) 3! (b) 2! + 4! (c) 2!× 3!

Solution : (a) $3! = 3\times 2\times 1 = 6$

(b) $2! = 2\times 1 = 2$

$4! = 4\times 3\times 2\times 1 = 24$

Therefore, $2! + 4! = 2+24 = 26$

(c) $2! \times 3! = 2\times6 = 12$

Notice that $n!$ satisfies the relation

$$n! = n\times(n-1)! \qquad \text{... (7.2)}$$

This is because, $n\,(n-1)! = n\,[(n-1)\,.\,(n-2)...\,2.1]$

$$= n\,.\,(n-1)\,.\,(n-2)\,...2.1$$

$$= n!$$

Of course, the above relation is valid only for $n \geq 2$ because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define

$$0! = 1 \qquad \text{... (7.3)}$$

then the relation 7.2 holds for $n = 1$ also.

Example 7.8 Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution : We have to arrange 6 books.

The number of permutations of n objects is $n! = n.\,(n-1)\,.\,(n-2)\,...\,2.1$

Here $n = 6$ and therefore, number of permutations is $6.5.4.3.2.1 = 720$

CHECK YOUR PROGRESS 7.3

1. (a) Evaluate : (i) 6! (ii) 7! (iii) $7! + 3!$ (iv) $6! \times 4!$ (v) $\dfrac{5!}{3!.2!}$

   (b) Which of the following statements are true?

   (i) $2! \times 3! = 6!$ (ii) $2! + 4! = 6!$

   (iii) 3! divides 4! (iv) $4! - 2! = 2!$

2. (a) 5 students are staying in a dormitory. In how many ways can you allot 5 beds to them?

   (b) In how many ways can the letters of the word 'TRIANGLE' be arranged?

   (c) How many four digit numbers can be formed with digits 1, 2, 3 and 4 and with distinct digits?

7.3 PERMUTATION OF r OBJECTS OUT OF n OBJECTS

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give any one of the five books to Asha and after that you can give any one of the remaining four books to Akhtar. After that, you can give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in $5 \times 4 \times 3$ *ie.*60 ways.

More generally, suppose you have to arrange r objects out of n objects. In how many ways can you do it? Let us view this in the following way. Suppose you have n objects and you have to arrange r of these in r boxes, one object in each box.

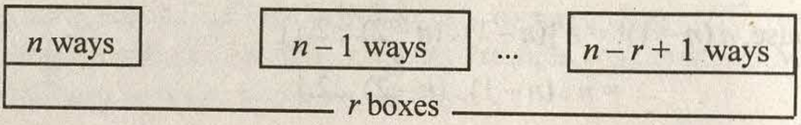


Fig. 7.1

Suppose there is one box. $r = 1$. You can put any of the n objects in it and this can be done in n ways. Suppose there are two boxes. $r = 2$. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining $n - 1$ objects. So, by the counting principle, the two boxes can be filled in $n(n-1)$ ways. Similarly, 3 boxes can be filled in $n(n-1)(n-2)$ ways.

In general, we have the following theorem.

Theorem 7.2 The number of permutations of r objects out of n objects is

$$n(n-1)\cdots(n-r+1).$$

The number of permutations of r objects out of n objects is usually denoted by ${}^nP_r$.

Thus,

$${}^nP_r = n(n-1)(n-2)..(n-r+1) \quad \text{... (7.4)}$$

Proof : Suppose we have to arrange r objects out of n different objects. In fact it is equivalent to filling r places, each with one of the objects out of the given n objects.

The first place can be filled in n different ways. Once this has been done, the second place can be filled by any one of the remaining $(n-1)$ objects, in $(n-1)$ ways. Similarly, the third place can be filled in $(n-2)$ ways and so on. The last place, the rth place can be filled in $[n-(r-1)]$ i.e. $(n-r+1)$ different ways. You may easily see, as to why this is so.

Using the Counting Principle, we get the required number of arrangements of r out of n objects is $n(n-1)(n-2)............(n-r+1)$

Example 7.9 Evaluate : (a) ${}^4P_2$ (b) ${}^6P_3$ (c) $\dfrac{{}^4P_3}{{}^3P_2}$ (d) ${}^6P_3 \times {}^5P_2$

Solution : (a) ${}^4P_2 = 4(4-1) = 4 \times 3 = 12.$

(b) ${}^6P_3 = 6(6-1)\,(6-2) = 6 \times 5 \times 4 = 120.$

(c) $$\frac{{}^4P_3}{{}^3P_2} = \frac{4(4-1)(4-2)}{3(3-1)} = \frac{4\times3\times2}{3\times2} = 4$$

(d) $${}^6P_3 \times {}^5P_2 = 6\,(6-1)\,(6-2)\times 5\,(5-1)$$

$$= 6\times5\times4\times5\times4 = 2400$$

Example 7.10 If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution : We have to find number of permutations of 4 objects out of 6 objects.

This number is ${}^6P_4 = 6(6-1)(6-2)(6-3) = 6.5.4.3 = 360$

Therefore, cards can be sent in 360 ways.

Consider the formula for ${}^nP_r$, namely, ${}^nP_r = n\,(n-1)\,...\,(n-r+1)$. This can be obtained by removing the terms $n-r, n-r-1, ..., 2, 1$ from the product for n!. The product of these terms is $(n-r)\,(n-r-1)\,...2.1$, i.e., $(n-r)!$.

$$\text{Now, } \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)(n-r)...2.1}{(n-r)\,(n-r-1)...2.1}$$

$$= n(n-1)(n-2)...(n-r+1)$$

$$= {}^nP_r$$

So, using the factorial notation, this formula can be written as follows :

$${}^nP_r = \frac{n!}{(n-r)!} \qquad ... (7.5)$$

Example 7.11 Find the value of ${}^nP_0$.

Solution : Here $r = 0$. Using relation 7.5, we get

$${}^nP_0 = \frac{n!}{n!} = 1$$

Example 7.12 Show that $(n+1)\,{}^nP_r = {}^{n+1}P_{r+1}$

Solution : $$(n+1)\,{}^nP_r = (n+1)\frac{n!}{(n-r)!} = \frac{(n+1)\,n!}{(n-r)!}$$

$$= \frac{(n+1)!}{[(n+1)-(r+1)]!} \text{ [writing } n-r \text{ as } [(n+1)-(r+1)]]$$

$$= {}^{n+1}P_{r+1} \qquad \text{(By definition)}$$

CHECK YOUR PROGRESS 7.4

1. (a) Evaluate : (i) ${}^4P_2$ (ii) ${}^6P_3$ (iii) $\frac{{}^4P_3}{{}^3P_2}$ (iv) ${}^6P_3 \times {}^5P_2$ (v) ${}^nP_n$

   (b) Check whether of the following statements are true or false :

   (i) $6 \times {}^5P_2 = {}^6P_2$ (ii) $4 \times {}^7P_3 = {}^7P_4$

   (iii) ${}^3P_2 \times {}^4P_2 = {}^{12}P_4$ (iv) ${}^3P_2 + {}^4P_2 = {}^7P_4$

2. (a) (i) What is the maximum possible number of 3- letter words in English that do not contain any vowel?

   (ii) What is the maximum possible number of 3- letter words in English which do not have any vowel other than 'a'?

   (b) Suppose you have 2 cots and 5 bedspreads in your house. In how many ways can you put the bedspreads on your cots?

   (c) You want to send Diwali Greetings to 4 friends and you have 7 greeting cards with you. In how many ways can you do it?

3. Show that ${}^nP_{n-1} = {}^nP_n$.

4. Show that $(n-r)\,{}^nP_r = {}^nP_{r+1}$.

7.4 PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 7.13 Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution : Let the beds be numbered 1 to 7.

Case 1 : Suppose Anju is allotted bed number 1.

Then, Parvin cannot be allotted bed number 2.

So Parvin can be allotted a bed in 5 ways.

After alloting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways. So, in this case the beds can be allotted in $5 \times 5!\,ways = 600$ ways.

Case 2 : Anju is allotted bed number 7.

Then, Parvin cannot be allotted bed number 6

As in Case 1, the beds can be allotted in 600 ways.

Case 3 : Anju is allotted one of the beds numbered 2,3,4,5 or 6.

Parvin cannot be allotted the beds on the right hand side and left hand side of Anju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin.

Therefore, Parvin can be allotted a bed in 4 ways in all these cases.

After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways.

Therefore, in each of these cases, the beds can be allotted in $4 \times 5! = 480$ ways.

$\therefore$ The beds can be allotted in

$(2 \times 600 + 5 \times 480)\, ways = (1200 + 2400)\, ways = 3600$ ways.

Example 7.14 In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution : They have to be arranged in the following way :

L	T	L	T	L	T	L	T	L

The 5 lions should be arranged in the 5 places marked 'L'.

This can be done in 5! ways.

The 4 tigers should be in the 4 places marked 'T'.

This can be done in 4! ways.

Therefore, the lions and the tigers can be arranged in $5! \times 4!\, ways = 2880$ ways.

Example 7.15 There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairytales, novels and plays.

Solution : There are 4 books on fairy tales and they have to be put together.

They can be arranged in 4! ways.
Similarly, there are 5 novels.
They can be arranged in 5! ways.
And there are 3 plays.

They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in $4! \times 5! \times 3!$ ways $= 17280$ ways.

Example 7.16 Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 7.15. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution : First, we consider the books on fairy tales, novels and plays as single objects.

These three objects can be arranged in 3!ways = 6 ways.

Let us fix one of these 6 arrangements.

This may give us a specific order, say, novels → fairy tales → plays.

Given this order, the books on the same subject can be arranged as follows:

The 5 novels can be arranged in 5! = 120 ways.

The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 3 plays can be arranged in 3! = 6 ways.

For a given order, the books can be arranged in $120 \times 24 \times 6 = 17280$ ways.

Therefore, for all the 6 possible orders the books can be arranged in $6 \times 17280 = 103680$ ways.

Example 7.17 In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution : Let 4 girls be one unit and now there are 6 units in all.

They can be arranged in 6! ways.

In each of these arrangements 4 girls can be arranged in 4! ways.

∴ Total number of arrangements in which girls are always together

$= 6! \times 4!$

$= 720 \times 24$

$= 17280$

Example 7.18 How many arrangements of the letters of the word 'BENGALI' can be made

(i) if the vowels are never together.

(ii) if the vowels are to occupy only odd places.

Solution : There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants.

(i) Considering vowels *a*, *e*, *i* as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.

∴ Total number of words $= 5! \times 3!$

$= 120 \times 6 = 720$

(ii) There are 4 odd places and 3 even places. 3 vowels can occupy 4 odd places in ${}^4P_3$ ways and 4 constants can be arranged in ${}^4P_4$ ways.

$\therefore$ Number of words $= {}^4P_3 \times {}^4P_4 = 24 \times 24$

$$= 576.$$

CHECK YOUR PROGRESS 7.5

1. Mr. Gupta with Ms. Gupta and their four children is travelling by train. Two lower berths, two middle berths and 2 upper berths have been allotted to them. Mr. Gupta has undergone a knee surgery and needs a lower berth while Ms. Gupta wants to rest during the journey and needs an upper berth. In how many ways can the berths be shared by the family?
2. Consider the word UNBIASED. How many words can be formed with the letters of the word in which no two vowels are together?
3. There are 4 books on Mathematics, 5 books on English and 6 books on Science. In how many ways can you arrange them so that books on the same subject are together and they are arranged in the order Mathematics → English → Science.
4. There are 3 Physics books, 4 Chemistry books, 5 Botany books and 3 Zoology books. In how many ways can you arrange them so that the books on the same subject are together?
5. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can this be done?
6. Find the number of permutations of the letters of the word 'TENDULKAR', in each of the following cases :

   (i) beginning with T and ending with R.

   (ii) vowels are always together.

   (iii) vowels are never together.

7.5 COMBINATIONS

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S_1, S_2, S_3, S_4. Then you can choose two pairs in the following ways :

1. $\{S_1, S_2\}$	2. $\{S_1, S_3\}$	3. $\{S_1, S_4\}$
4. $\{S_2, S_3\}$	5. $\{S_2, S_4\}$	6. $\{S_3, S_4\}$

[Observe that $\{S_1, S_2\}$ is the same as $\{S_2, S_1\}$]. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know from section 7.3, that it can be done in ${}^4P_2 = 12$ ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below :

1. $\{S_1, S_2\} \rightarrow S_1$ on Monday and S_2 on Tuesday or S_2 on Monday and S_1 on Tuesday
2. $\{S_1, S_3\} \rightarrow S_1$ on Monday and S_3 on Tuesday or S_3 on Monday and S_1 on Tuesday
3. $\{S_1, S_4\} \rightarrow S_1$ on Monday and S_4 on Tuesday or S_4 on Monday and S_1 on Tuesday
4. $\{S_2, S_3\} \rightarrow S_2$ on Monday and S_3 on Tuesday or S_3 on Monday and S_2 on Tuesday
5. $\{S_2, S_4\} \rightarrow S_2$ on Monday and S_4 on Tuesday or S_4 on Monday and S_2 on Tuesday
6. $\{S_3, S_4\} \rightarrow S_3$ on Monday and S_4 on Tuesday or S_4 on Monday and S_3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.

This argument holds good in general as we can see from the following theorem.

Theorem 7.3 Let $n \geq 1$ be an integer and $r \leq n$. Let us denote the number of ways of choosing *r* objects out of *n* objects by ${}^nC_r$. Then

$$ {}^nC_r = \frac{{}^nP_r}{r!} \qquad \text{... (7.6)} $$

Proof : We can choose *r* objects out of *n* objects in ${}^nC_r$ ways. Each of the *r* objects chosen can be arranged in *r*! ways. The number of ways of arranging *r* objects is *r*!. Thus, by the counting principle, the number of ways of choosing *r* objects and arranging the *r* objects chosen can be done in ${}^nC_r r!$ ways. But, this is precisely ${}^nP_r$. In other words, we have

$$ {}^nP_r = r!.{}^nC_r \qquad \text{... (7.7)} $$

Dividing both sides by *r*!, we get the result in the theorem.

Here is an example to help you to familiarise yourself with ${}^nC_r$.

Example 7.19 Evaluate each of the following :

(a) ${}^5C_2$ (b) ${}^5C_3$

(c) ${}^4C_3 + {}^4C_2$ (d) $\dfrac{{}^6C_3}{{}^4C_2}$

Solution : (a) ${}^5C_2 = \frac{{}^5P_2}{2!} = \frac{5.4}{1.2} = 10$. (b) ${}^5C_3 = \frac{{}^5P_3}{3!} = \frac{5.4.3}{1.2.3} = 10$.

(c) $^4C_3 + {}^4C_2 = \frac{^4P_3}{3!} + \frac{^4P_2}{2!} = \frac{4.3.2}{1.2.3} + \frac{4.3}{1.2} = 4 + 6 = 10$

(d) $^6C_3 = \frac{^6P_3}{3!} = \frac{6.5.4}{1.2.3} = 20$ and $^4C_2 = \frac{4.3}{1.2} = 6$

$$\therefore \frac{^6C_3}{^4C_2} = \frac{20}{6} = \frac{10}{3}.$$

Example 7.20 Find the number of subsets of the set $\{1,2,3,4,5,6,7,8,9,10,11\}$ having 4 elements.

Solution : Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

By relation (7.6), this can be done in

$$^{11}C_4 = \frac{11.10.9.8}{1.2.3.4} = 330 \text{ ways.}$$

Example 7.21 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution : For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is $^{12}C_4 = 495$. Therefore, we can draw 495 quadrilaterals.

Example 7.22 In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution : Number of ways of choosing 2 black pens from 5 black pens

$$= {}^5C_2 = \frac{^5P_2}{2!} = \frac{5.4}{1.2} = 10.$$

Number of ways of choosing 2 white pens from 3 white pens

$$= {}^3C_2 = \frac{^3P_2}{2!} = \frac{3.2}{1.2} = 3.$$

Number of ways of choosing 2 red pens from 4 red pens

$$= {}^4C_2 = \frac{^4P_2}{2!} = \frac{4.3}{1.2} = 6.$$

$\therefore$ By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in $10 \times 3 \times 6 = 180$ ways.

Example 7.23 A question paper consists of 10 questions divided into two parts *A* and *B*. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part *A* and at least 2 from part *B*. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution : The candidate has to select six questions in all of which at least two should be from Part *A* and two should be from Part *B*. He can select questions in any of the following ways :

	Part *A*	Part *B*
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is

$${}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$$

If the candidate follows choice (ii), the number of ways in which he can do so is

$${}^5C_3 \times {}^5C_3 = 10 \times 10 = 100.$$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is

$${}^5C_4 \times {}^5C_2 = 50.$$

Therefore, the candidate can select the question in $50 + 100 + 50 = 200$ ways.

Example 7.24 A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when

(i) at least 2 women are included?

(ii) atmost 2 women are included?

Solution : (i) When at least 2 women are included.

The committee may consist of

3 women, 2 men : It can be done in ${}^4C_3 \times {}^6C_2$ ways.

or, 4 women, 1 man : It can be done in ${}^4C_4 \times {}^6C_1$ ways.

or, 2 women, 3 men : It can be done in ${}^4C_2 \times {}^6C_3$ ways.

$\therefore$ Total number of ways of forming the committee

$$= {}^4C_2 . {}^6C_3 + {}^4C_3 . {}^6C_2 + {}^4C_4 . {}^6C_1$$

$$= 6 \times 20 + 4 \times 15 + 1 \times 6$$

$$= 120 + 60 + 6 = 186$$

(ii) When atmost 2 women are included

The committee may consist of

2 women, 3 men : It can be done in ${}^4C_2.{}^6C_3$ ways

or, 1 woman, 4 men : It can be done in ${}^4C_1.{}^6C_4$ ways

or, 5 men : It can be done in ${}^6C_5$ ways

∴ Total number of ways of forming the committee

$$= {}^4C_2.{}^6C_3 + {}^4C_1.{}^6C_4 + {}^6C_5$$

$$= 6 \times 20 + 4 \times 15 + 6$$

$$= 120 + 60 + 6 = 186$$

Example 7.25 The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and atleast 4 bowlers?

Solution : We are to choose 11 players including 1 wicket keeper and 4 bowlers

or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players

$$= {}^2C_1.{}^5C_4.{}^9C_6$$

$$= 2 \times \frac{5 \times 4 \times 3 \times 2.1}{4.3.2.1.} \times \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 2 \times 5 \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 840$$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players

$$= {}^2C_1.{}^5C_5.{}^9C_5$$

$$= 2 \times 1 \times \frac{9 \times 8 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2 \times 1} = 2 \times 1 \times \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 252$$

∴ Total number of ways of selecting the team

$$= 840 + 252 = 1092$$

CHECK YOUR PROGRESS 7.6

1. (a) Evaluate :

   (i) ${}^{13}C_3$ (ii) ${}^{9}C_5$ (iii) ${}^{8}C_2 + {}^{8}C_3$ (iv) $\frac{{}^{9}C_3}{{}^{6}C_3}$

   (b) Verify each of the following statement :

   (i) ${}^{5}C_2 = {}^{5}C_3$ (ii) ${}^{4}C_3 \times {}^{3}C_2 = {}^{12}C_6$

   (iii) ${}^{4}C_2 + {}^{4}C_3 = {}^{8}C_5$ (iv) ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3$

2. Find the number of subsets of the set {1, 3, 5, 7, 9, 11, 13, ..., 23}each having 3 elements.

3. There are 14 points lying on a circle. How many pentagons can be drawn using these points?

4. In a fruit basket there are 5 apples, 7 plums and 11 oranges. You have to pick 3 fruits of each type. In how many ways can you make your choice?

5. A question paper consists of 12 questions divided into two parts A and B, containing 5 and 7 questions repectively. A student is required to attempt 6 questions in all, selecting at least 2 from each part. In how many ways can a student select a question?

6. Out of 5 men and 3 women, a committee of 3 persons is to be formed. In how many ways can it be formed selecting (i) exactly 1 woman. (ii) atleast 1 woman.

7. A cricket team consists of 17 players. It includes 2 wicket keepers and 4 bowlers. In how many ways can a playing eleven be selected if we have to select 1 wicket keeper and atleast 3 bowlers?

8. To fill up 5 vacancies, 25 applications were recieved. There were 7 S.C. and 8 O.B.C. candidates among the applicants. If 2 posts were reserved for S.C. and 1 for O.B.C. candidates, find the number of ways in which selection could be made?

7.6 SOME SIMPLE PROPERTIES OF ${}^{n}C_r$

In this section we will prove some simple properties of ${}^{n}C_r$ which will make the computations of these coefficients simpler. Let us go back again to Theorem 7.3. Using relation 7.7 we can rewrite the formula for ${}^{n}C_r$ as follows :

$$ {}^{n}C_r = \frac{n!}{r!(n-r)!} \qquad \text{....(7.8)} $$

Example 7.26 Find the value of $^nC_0$.

Solution : Here $r = 0$. Therefore, $^nC_0 = \frac{n!}{0!n!} = \frac{1}{0!} = 1$,

since we have defined $0! = 1$.

The formula given in Theorem 7.3 was used in the previous section. As we will see shortly, the formula given in Equation 7.8 will be useful for proving certain properties of ^nC_r.

$$^nC_r = {^nC_{n-r}} \quad \text{...(7.9)}$$

This means just that the number of ways of choosing r objects out of n objects is the same as the number of ways of not choosing $(n-r)$ objects out of n objects. In the example described in the introduction, it just means that the number of ways of selecting 2 sets of dresses is the same as the number of ways of rejecting $4-2=2$ dresses. In Example 7.20, this means that the number of ways of choosing subsets with 4 elements is the same as the number of ways of rejecting 8 elements since choosing a particular subset of 4 elements is equivalent to rejecting its complement, which has 8 elements.

Let us now prove this relation using Equation 7.8. The denominator of the right hand side of this equation is $r!\ (n-r)!$. This does not change when we replace r by $n-r$.

$$(n-r)!.[n-(n-r)]! = (n-r)!.r!$$

The numerator is independent of r. Therefore, replacing r by $n-r$ in Equation 7.8 we get result.

How is the relation 7.9 useful? Using this formula, we get, for example, $^{100}C_{98}$ is the same as $^{100}C_2$. The second value is much more easier to calculate than the first one.

Example 7.27 Evaluate :

(a) $^7C_5$ (c) $^{10}C_9$

(b) $^{11}C_9$ (d) $^{12}C_9$

Solution : (a) From relation 7.9, we have

$$^7C_5 = {^7C_{7-5}} = {^7C_2} = \frac{7.6}{1.2} = 21$$

(b) Similarly $^{10}C_9 = {^{10}C_{10-9}} = {^{10}C_1} = 10$

(c) $^{11}C_9 = {^{11}C_{11-9}} = {^{11}C_2} = \frac{11.10}{1.2} = 55$

(d) $^{12}C_{10} = {^{12}C_{12-10}} = {^{12}C_2} = \frac{12.11}{1.2} = 66$

There is another relation satisfied by ${}^nC_r$ which is also useful. We have the following relation:

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r \qquad \text{....(7.10)} $$

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} $$

$$ = \frac{(n-1)!}{(n-r)(n-r-1)!(r-1)!} + \frac{(n-1)!}{r(n-r-1)!(r-1)!} $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right] $$

$$ = \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{n}{(n-r)r}\right] $$

$$ = \frac{n(n-1)!}{(n-r)(n-r-1)!r(r-1)!} $$

$$ = \frac{n!}{(n-r)!r!} = {}^nC_r $$

Example 7.28 Evaluate :

(a) ${}^6C_2 + {}^6C_1$ (b) ${}^8C_2 + {}^8C_1$

(c) ${}^5C_3 + {}^5C_2$ (d) ${}^{10}C_2 + {}^{10}C_3$

Solution : (a) Let us use relation (7.10) with $n = 7, r = 2$. So, ${}^6C_2 + {}^6C_1 = {}^7C_2 = 21$

(b) Here $n = 9,\ r = 2$. Therefore, ${}^8C_2 + {}^8C_1 = {}^9C_2 = 36$

(c) Here $n = 6,\ r = 3$. Therefore, ${}^5C_3 + {}^5C_2 = {}^6C_3 = 20$

(d) Here $n = 11, r = 3$. Therefore, ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3 = 165$

To understand the relation 7.10 better, let us go back to Example 7.20 In this example let us calculate the number of subsets of the set, $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. We can subdivide them into two kinds, those that contain a particular element, say 1, and those that do not contain 1. The number of subsets of the set having 4 elements and containing 1 is the same as the number of subsets of $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ having 3 elements. There are ${}^{10}C_3$ such subsets.

The number of subsets of the set having 4 elements and not containing 1 is the same as the number of subsets of the set {2,3,4,5,6,7,8,9,10,11,} having 4 elements. This is ${}^{10}C_4$. So, the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ having four elements is ${}^{10}C_3 + {}^{10}C_4$. But, this is ${}^{11}C_4$ as we have seen in the example. So, ${}^{10}C_3 + {}^{10}C_4 = {}^{11}C_4$. The same argument works for the number of r–element subsets of a set with n elements.

This reletion was noticed by the French Mathematician **Blaise Pascal** and was used in the so called **Pascal Triangle,** given below.

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The first row consists of single element ${}^0C_0 = 1$. The second row consists of ${}^1C_0$ and ${}^1C_1$. From the third row onwards, the rule for writing the entries is as follows. Add consecutive elements in the previous row and write the answer between the two entries. After exhausting all possible pairs, put the number 1 at the begining and the end of the row. For example, the third row is got as follows. Second row contains only two elements and they add up to 2. Now, put 1 before and after 2. For the fourth row, we have $1 + 2 = 3$, $2 + 1 = 3$. Then, we put $1 + 2 = 3$, $2 + 1 = 3$. Then we put 1 at the beginning and the end. Here, we are able to calculate, for example, ${}^3C_1$, ${}^3C_2$,, from ${}^2C_0$, ${}^2C_1$, ${}^2C_2$ by using the relation 7.10. The important thing is we are able to do it using addition alone.

The numbers ${}^nC_r$ occur as coefficents in the binomial expansions, and therefore, they are also called **binomial coefficents** as we will see in lesson 8. In particular, the property 7.10 will be used in the proof of the binomial theorem.

Example 7.29 If ${}^nC_{10} = {}^nC_{12}$ find n,

Solution : Using ${}^nC_r = {}^nC_{n-r}$ we get

$$n - 10 = 12$$

or, $n = 12 + 10 = 22$

CHECK YOUR PROGRESS 7.7

1. (a) Find the value of ${}^nC_{n-1}$. Is ${}^nC_{n-1} = {}^nC_n$? (b) Show that ${}^nC_n = {}^nC_0$

2. Evaluate :

   (a) ${}^9C_5$ (b) ${}^{14}C_{10}$

   (c) ${}^{13}C_9$ (d) ${}^{15}C_{12}$

3. Evaluate :

   (a) ${}^7C_3 + {}^7C_2$ (b) ${}^8C_4 + {}^8C_5$

   (c) ${}^9C_3 + {}^9C_2$ (d) ${}^{12}C_3 + {}^{12}C_2$

4. If ${}^{10}C_r = {}^{10}C_{2r+1}$, find the value of r. 5. If ${}^{18}C_r = {}^{18}C_{r+2}$ find ${}^rC_5$

PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 7.30 There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf?

Soluton : 4 novels can be selected out of 5 in ${}^5C_4$ ways. 2 biographies can be selected out of 4 in ${}^4C_2$ ways.

Number of ways of selecting novels and biographies

$$= {}^5C_4 \times {}^4C_2 = 5 \times 6 = 30$$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in $6! = 720$ ways.

By the Counting Principle, the total number of arrangements $= 30 \times 720 = 21600$

Example 7.31 From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels ?

Solution : From 5 consonants, 3 consonants can be selected in ${}^5C_3$ ways.

From 4 vowels, 2 vowels can be selected in ${}^4C_2$ ways.

Now with every selection, number of ways of arranging 5 letters is ${}^5P_5$

$\therefore$ Total number of words $= {}^5C_3 \times {}^4C_2 \times {}^5P_5$

$$= \frac{5\times4}{2\times1}\times\frac{4\times3}{2\times1}\times5!$$

$$= 10\times6\times5\times4\times3\times2\times1 = 7200$$

CHECK YOUR PROGRESS 7.8

1. There are 5 Mathematics, 4 Physics and 5 Chemistry books. In how many ways can you arrange 4 Mathematics, 3 Physics and 4 Chemistry books.

   (a) if the books on the same subjects are arranged together, but the order in which the books are arranged within a subject doesn't matter ?

   (b) if books on the same subjects are arranged together and the order in which books are arranged within subject matters ?

2. There are 9 consonants and 5 vowels. How many words of 7 letters can be formed using 4 consonents and 3 vowels ?

3. In how many ways can you invite at least one of your six friends to a dinner?

4. In an examination, an examinee is required to pass in four different subjects. In how many ways can he fail?

LET US SUM UP

- Fundamental principle of counting states.

  If there are n events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the n events can occur in

  $m_1 \times m_2 \times m_3 \times \ldots\ldots\ldots \times m_{n-1} \times m_n$ ways.

- The number of permutations of n objects taken all at a time is $n!$

- ${}^nP_r = \dfrac{n!}{(n-r)!}$

- ${}^nP_n = n!$

- The number of ways of selecting r objects out of n objects ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

- ${}^nC_r = {}^nC_{n-r}$

- ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. There are 8 true - false questions in an examination. How many responses are possible ?
2. The six faces of a die are numbered 1,2,3,4,5 and 6. Two such dice are thrown simultaneously. In how many ways can they turn up ?
3. A restaurant has 3 vegetables, 2 salads and 2 types of bread. If a customer wants 1 vegetable, 1 salad and 1 bread, how many choices does he have ?
4. Suppose you want to paper your walls. Wall papers are available in 4 diffrent backgrounds colours with 7 different designs of 5 different colours on them. In how many ways can you select your wall paper ?
5. In how many ways can 7 students be seated in a row on 7 seats ?
6. Determine the number of 8 letter words that can be formed from the letters of the word *ALTRUISM*.
7. If you have 5 windows and 8 curtains in your house, in how many ways can you put the curtains on the windows ?
8. Determine the maximum number of 3- letter words that can be formed from the letters of the word *POLICY*.
9. There are 10 athletes participating in a race and there are three prizes, 1st, 2nd and 3rd to be awarded. In how many ways can these be awarded ?
10. In how many ways can you arrange the letters of the word *ATTAIN* so that the *T*s are together?
11. A group of 12 friends meet at a party. Each person shake hands once with all others. How many hand shakes will be there. ?
12. Suppose that you own a shop which sells televisions. You are selling 5 different kinds of television sets, but your show case has enough space for display of 3 televison sets only. In how many ways can you select the television sets for the display ?
13. A contractor needs 4 carpenters. Five equally qualified carpenters apply for the job. In how many ways can the contractor make the selection ?
14. In how many ways can a committe of 9 can be selected from a group of 13?
15. In how many ways can a committee of 3 men and 2 women be selected from a group of 15 men and 12 women ?

16. In how ways can 6 persons be selected from 4 grade 1 and 7 grade II officers, so as to include at least two officers from each category ?

17. Out of 6 boys and 4 girls, a committee of 5 has to be formed. In how many ways can this be done if we take :

    (a) exactly 2 girls?

    (b) at least 2 girls?

18. The English alphabet has 5 vowels and 21 consonants. What is the maximum number of words, that can be formed from the alphabet with 2 different vowels and 2 different consonants?

19. From 5 consonants and 5 vowels, how many words can be formed using 3 consonants and 2 vowels?

20. In a school annual day function a variety programme was organised. It was planned that there would be 3 short plays, 6 recitals and 4 dance programmes. However, the chief guest invited for the function took much longer time than expected to finish his speech. To finish in time, it was decided that only 2 short plays, 4 recitals and 3 dance programmes would be perfomed, How many choices were available to them ?

    (a) if the programmes can be perfomed in any order ?

    (b) if the programmes of the same kind were perfomed at a stretch?

    (c) if the programmes of the same kind were perfomed at a strech and considering the order of performance of the programmes of the same kind ?

ANSWERS

CHECK YOUR PROGRESS 7.1

1. (a) 180 (b) 8 (c) 12 (d) 20

2. (a) 48 (b) 36

CHECK YOUR PROGRESS 7.2

1. (a) 17576 (b) 900

2. (a) 105 (b) 60

3. (a) 24 (b) 24

CHECK YOUR PROGRESS 7.3

1. (a) (i) 720 (ii) 5040 (iii) 5046 (iv) 17280 (v) 10

(b) (i) False (ii) False (iii) True (iv) False

2. (a) 120 (b) 40320 (c) 24

CHECK YOUR PROGRESS 7.4

1. (a) (i) 12 (ii) 120 (iii) 4 (iv) 7200 (v) $n!$

(b) (i) False (ii) True (iii) False (iv) False

2. (a) (i) 7980 (ii) 9240 (b) 20 (c) 840

CHECK YOUR PROGRESS 7.5

1. 96 2. 1152 3. 2073600 4. 2488320

5. 144 6. (i) 5040 (ii) 30240 (iii) 332640

CHECK YOUR PROGRESS 7.6

1. (a) (i) 286

(ii) 126

(iii) 84

(iv) $\frac{21}{5}$

(b) (i) True

(ii) False

(iii) False

(iv) True

2. 1771

3. 2002

4. 57750

5. 805

6. (i) 30

   (ii) 46

7. 3564

8. 7560

CHECK YOUR PROGRESS 7.7

1. (a) *n* , No

2. (a) 126
   (b) 1001
   (c) 715
   (d) 455

3. (a) 56
   (b) 126
   (c) 120
   (d) 286

4. 3

5. 56

CHECK YOUR PROGRESS 7.8

1 (a) 600
  (b) 207 3600

2. 6350400

3. 63

4. 15

TERMINAL EXERCISE

1. 256
2. 36
3. 12
4. 140
5. 5040
6. 40320
7. 6720
8. 120
9. 720
10. 120
11. 66
12. 10
13. 5
14. 715
15. 30030
16. 371
17. (a) 120
    (b) 186
18. 50400
19. 12000
20. (a) 65318400
    (b) 1080
    (c) 311040

8

BINOMIAL THEOREM

Suppose you need to calculate the amount of interest you will get after 5 years on a sum of money that you have invested at the rate of 15% compound interest per year. Or suppose we need to find the size of the population of a country after 10 years if we know the annual growth rate. A result that will help in finding these quantities is the **binomial theorem.** This theorem, as you will see, helps us to calculate the rational powers of any real binomial expression, that is, any expression involving two terms.

The binomial theorem, was known to Indian and Greek mathematicians in the 3rd century B.C. for some cases. The credit for the result for natural exponents goes to the Arab poet and mathematician Omar Khayyam (A.D. 1048-1122). Further generalisation to rational exponents was done by the British mathematician Newton (A.D. 1642-1727).

There was a reason for looking for further generalisation, apart from mathematical interest. The reason was its many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus, and in approximating numbers like $(1.02)^7$, $3^{1/5}$, etc. We shall discuss a few of them in this lesson. Before discussing Binomial Theorem, we shall introduce the concept of Principle of Mathematical Induction, which we shall be using in proving the Binomial Theorem for a positive integral index. This principle is also useful in making generalisations from particular statements/results.

OBJECTIVES

After studying this lesson, you will be able to:

- state the Principle of (finite) Mathematical Induction;
- verify the truth or otherwise of the statement P(n) for n = 1;
- verify if P(k+1) is true, assuming that P(k) is true;
- use principle of mathematical induction to establish the truth or otherwise of mathematical statements;

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x+y)^n$ for different values of x and y using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;
- write the binomial expansion for negative as well as for rational indices;
- apply the binomial expansion for finding approximate values of numbers like $\sqrt[3]{9}, \sqrt{2}, \sqrt[3]{3}$ etc; and
- apply the binomial expansion to evaluate algebraic expressions like $\left(3-\frac{5}{x}\right)^7$, where x is so small that x^2, and higher powers of x can be neglected.

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

8.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true** or **false** are called **statement** or **propositions.** For instance,

"I am 20 years old" and "If $x = 3$, then $x^2 = 9$" are statements, but 'When will you leave ?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. For example, '$x - 5 = 7$' is not a statement, because we don't know what x, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number x' are statements, the first one true and the second one false.

Example 8.1 Which of the following sentences is a statement ?

(i) India has never had a woman President.

(ii) 5 is an even number.

(iii) $x^n > 1$

(iv) $(a+b)^2 = a^2 + 2ab + b^2$

Solution : (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x and y can take.

Now look at (iv). At first glance , you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identity. Therefore, in this case, even though we have not specified the range of values for a and b, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement given below :

$$1+2+...+n=\frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement P (n) [P stands for proposition].

Then P (1) would be $1=\frac{1(1+1)}{2}$

Similarly, P (2) would be the statement

$$1+2=\frac{2(2+1)}{2} \text{ and so on.}$$

Let us look at some examples to help you get used to this notation.

Example 8.2 If P (n) denotes $2^n > n-1$, write P (1), P (k) and P (k+1), where $k \in N$.

Solution : Replacing n by 1, k and k + 1, respectively in P (n), we get

$P(1): 2^1 > 2-1$, i.e., $2 > 1$

$P(k): 2^k > k-1$

$P(k+1): 2^{k+1} > (k+1)-1$, i.e., $2^{k+1} > k$

Example 8.3 If P (n) is the statement

$$‘1+4+7+(3n-2)=\frac{n(3n-1)}{2}$$

write $P(1)$, $P(k)$ and $P(k+1)$.

Solution : To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till $3\times1-2$, i.e., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS) of $P(1)=\frac{1\times(3\times1-1)}{2}=1$

Therefore, $P(1)$ is $1 = 1$.

Replacing n by 2, we get

$$P(2) : 1 + 4 = \frac{2 \times (3 \times 2 - 1)}{2}, \text{ i.e., } 5 = 5.$$

Replacing n by k and $k+1$, respectively, we get

$$P(k) : 1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k - 1)}{2}$$

$$P(k + 1) : 1 + 4 + 7 + ... + (3k - 2) + [3\,(k + 1) - 2]$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

$$\text{i.e. , } 1 + 4 + 7 + ... + (3k + 1) = \frac{(k + 1)[(3k + 2)}{2}$$

CHECK YOUR PROGRESS 8.1

1. Determine which of the following are statements :

(a) $1 + 2 + 4 \;\; .... + 2^n > 20$ (b) $1 + 2 + 3 + ... + 10 = 99$

(c) Chennai is much nicer than Mumbai. (d) Where is Taj Mahal ?

(e) $\frac{1}{1\times 2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n = 5$ (f) $\operatorname{cosec}\theta < 1$

2. Given that $P(n)$: 6 is a factor of $n^3 + 5n$, write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$ where k is a natural number.

3. Write $P(1)$, $P(k)$ and $P(k + 1)$, if $P(n)$ is:

(a) $2^n \geq n + 1$ (b) $(1 + x)^n \geq 1 + nx$

(c) $n\,(n+1)\,(n+2)$ is divisible by 6. (d) $(x^n - y^n)$ is divisible by $(x - y)$.

(e) $(ab)^n = a^n\, b^n$ (f) $\left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right)$ is a natural number.

4. Write $P(1)$, $P(2)$, $P(k)$ and $P(k+1)$, if $P(n)$ is :

(a) $\frac{1}{1\times 2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

(b) $1 + 3 + 5 + ... + (2n - 1) = n^2$

(c) $(1\times 2)+(2\times 3)+....+n(n+1) < n(n+1)^2$

(d) $\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+...\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$

Now ,when you are given a statement like the ones given in Examples 2 and 3, how would you check whether it is true or false ? One effective method is mathematical induction, which we shall now discuss.

8.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friend just had a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of **deductive** logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations". In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to. The following principle allows us to check if this happens.

The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k+1)$ is true; then $P(n)$ must be true for every natural number n.

***Note** that condition (ii) above **does not** say that P(k) **is** true. It says that **whenever** P(k) is true, then P(k + 1) is true'.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1+1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2+1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. It is now clear that if we continue like this, we shall get that $P(11)$ is true.

It is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Let us now see, through examples, how we can apply the principle of mathematical induction to prove various types of mathematical statements.

Example 8.4 Prove that

$$1+2+3+\cdots+n=\frac{n}{2}(n+1), \text{ where } n \text{ is a natural number.}$$

Solution: We have

$$P(n) : 1+2+3+\ldots+n=\frac{n}{2}(n+1)$$

Therefore, $P(1)$ is '$1=\frac{1}{2}(1+1)$', which is true,.

Therefore, $P(1)$ is true.

Let us now see, if $P(k+1)$ is true whenever $P(k)$ is true.

Let us, therefore, assume that $P(k)$ is true, i.e.,

$$1+2+3 \ldots +k=\frac{k}{2}(k+1) \qquad \text{....(i)}$$

Now, $P(k+1)$ is $1+2+3+\ldots+k+(k+1)=\frac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $P(k+1)=(1+2+3 \ldots +k)+(k+1)$

$$=\frac{k}{2}(k+1)+(k+1) \qquad \text{....[From (i)]}$$

$$=(k+1)\left(\frac{k}{2}+1\right)$$

$$=\frac{(k+1)(k+2)}{2}$$

$$= \text{RHS of } P(k+1)$$

So, $P(k+1)$ is true, if we assume that $P(k)$ is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps – the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

Example 8.5 Prove that

$$1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \dots + n.2^n = (n-1).2^{n+1} + 2,$$

where n is a natural number.

Solution : Here $P(n)$ is $1.2^1 + 2.2^2 + 3.2^3 + \dots + n.2^n = (n-1)\,2^{n+1} + 2$

Therefore, $P(1)$ is $1.2^1 = (1-1)2^{1+1} + 2$, i.e., $2 = 2$.

So, $P(1)$ is true.

We assume that $P(k)$ is true, i.e.,

$$1.2^1 + 2.2^2 + 3.2^3 + \dots + k.2^k = (k-1)2^{k+1} + 2 \qquad \text{....(i)}$$

Now will prove that $P(k+1)$ is true.

Now $P(k+1)$ is

$$1.2^1 + 2.2^2 + 3.2^3 + \dots + k.2^k + (k+1)2^{k+1} = [(k+1)-1]2^{(k+1+1)} + 2$$

$$= k.2^{k+2} + 2$$

$$\text{LHS of } P(k+1) = (1.2^1 + 2.2^2 + 3.2^3 + \dots. + k.2k + (k+1)\,2^{k+1}$$

$$= 2^{k+1}\left[(k-1)+(k+1)\right]+2$$

$$= 2^{k+1}(2k)+2 \qquad \text{[Using (i)]}$$

$$= k2^{k+2}+2$$

$$= \text{RHS of } P(k+1)$$

Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 8.6 For ever natural number n, prove that $\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x+y)$, where $x, y \in N$.

Solution: Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement

'$\left(x^{2n-1} + y^{2n-1}\right)$ is divisible by $(x+y)$',

Then $P(1)$ is '$\left(x^{2-1} + y^{2-1}\right)$ is divisible by $(x+y)$', i.e., '$(x+y)$ is divisible by $(x+y)$', which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number k, i.e., $(x^{2k-1} + y^{2k-1})$ is divisible by $(x + y)$.

This means that for some natural number t, $x^{2k-1} + y^{2k-1} = (x+y)t$

Then, $x^{2k-1} = (x+y)t - y^{2k-1}$

We wish to prove that P $(k+1)$ is true, i.e., '$[x^{2(k+1)-1} + y^{2(k+1)-1}]$ is divisible by $(x+y)$' is true.

Now,

$$\begin{aligned}
x^{2(k+1)-1} + y^{2(k+1)-1} &= x^{2k+1} + y^{2k+1} \\
&= x^{2k-1+2} + y^{2k+1} \\
&= x^{2} \cdot x^{2k-1} + y^{2k+1} \\
&= x^{2}[(x+y)t - y^{2k-1}] + y^{2k+1} \qquad \text{[From (1)]} \\
&= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2k+1} \\
&= x^{2}(x+y)t - x^{2}y^{2k-1} + y^{2}y^{2k-1} \\
&= x^{2}(x+y)t - y^{2k-1}(x^{2} - y^{2}) \\
&= (x+y)[x^{2}t - (x-y)y^{2k-1}]
\end{aligned}$$

which is divisible by of $(x+y)$.

Thus, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 8.7 Prove that $2^n > n$ for every natural number n.

Solution: We have $P(n) : 2^n > n$.

Therefore, $P(1) : 2^1 > 1$, i.e., $2 > 1$, which is true.

We assume $P(k)$ to be true, that is,

$$2^k > k \qquad \text{... (i)}$$

We wish to prove that $P(k + 1)$ is true, i.e. $2^{k+1} > k + 1$.

Now, multiplying both sides of (i) by 2, we get

$$2^{k+1} > 2k$$

$$\Rightarrow 2^{k+1} > k + 1, \text{ since } k > 1.$$

Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say a (as in Example 8.8 below). In such a situation, we replace $P(1)$ by $P(a+1)$ in the statement of the principle.

Example 8.8 Prove that

$$n^2 > 2(n+1) \text{ for all } n \geq 3, \text{ where } n \text{ is a natural number.}$$

Solution: For $n \geq 3$, let us call the following statement

$$P(n) : n^2 > 2(n+1)$$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$. We, therefore, see whether $P(3)$ is true.

$P(3) : 3^2 > 2 \times 4$, i.e. $9 > 8$.

So, $P(3)$ is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is

$$k^2 > 2(k+1) \qquad \text{........ (i)}$$

We wish to prove that $P(k+1)$ is true.

$P(k+1) : (k+1)^2 > 2(k+2)$

LHS of $P(k+1) = (k+1)^2$

$= k^2 + 2k + 1$

$> 2(k+1) + 2k + 1$ ... [By (i)]

$> 3 + 2k + 1$, since $2(k+1) > 3$.

$= 2(k+2)$,

Thus, $(k+1)^2 > 2(k+2)$

Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Example 8.9 Using principle of mathematical induction, prove that

$$\left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right) \text{ is a natural number for all natural numbers } n.$$

Solution :

Let $P(n) : \left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right)$ be a natural number.

$\therefore P(1): \left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$ is a natural number.

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1$, which is a natural number.

$\therefore$ $P(1)$ is true.

Let $P(k): \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ... (i)

Now $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$=\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+1 \quad \text{...(ii)}$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

$\therefore$ (ii) being sum of natural numbers is a natural number.

$\therefore$ $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(n)$ is true for all natural numbers n.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

CHECK YOUR PROGRESS 8.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number n:

   (a) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n}{6}(n+1)(2n+1)$

   (b) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = (1+2+\ldots\ldots+n)^2$

   (c) $1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$

   (d) $1+4+7+\ldots\ldots+(3n-2) = \frac{n}{2}(3n-1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number n:

   (a) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (b) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

   (c) $(1\times 2) + (2\times 3) + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

3. For every natural number n, prove that

   (a) $n^3 + 5n$ is divisible by 6. (b) $(x^n - 1)$ is divisible by $(x-1)$.

   (c) $(n^3 + 2n)$ is divisible by 3. (d) 4 divides $(n^4 + 2n^3 + n^2)$.

4. Prove the following inequalities for any natural number n:

   (a) $3^n \geq 2n + 1$ (b) $4^{2n} > 15n$

   (c) $1+2+\ldots+n < \frac{1}{8}(2n+1)^2$

5. Prove the following statements using induction:

   (a) $2^n > n^2$ for $n \geq 5$, where n *is any natural number.*

   (b) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$ for any natural number n greater than 1.

6. Prove that $n(n^2 - 1)$ is divisible by 3 for every natural number n greater than 1.

To prove that a statement $P(n)$ is true for every $n \in N$, both the basic as well as the induction steps must hold.

If even one of these conditions does not hold, then the proof is invalid. For instance, if $P(n)$ is' $(a+b)^n \leq a^n + b^n$' for all reals a and b, then $P(1)$ is certainly true. But, $P(k)$ being true does not imply the truth of $P(k+1)$. So, the statement is not true for every natural number n. (For instance, $(2+3)^2 \neq 2^2+3^2$).

Notes

As another example, take $P(n)$ to be $n > \frac{n}{2} + 20$.

In this case, $P(1)$ is not true. But the induction step is true. Since $P(k)$ being true.

$$\Rightarrow k > \frac{k}{2} + 20$$

$$\mathrm{P(k+1)} : \mathrm{k} + 1 > \frac{\mathrm{k}}{2} + 20 + 1 > \frac{\mathrm{k}}{2} + 20 + \frac{1}{2} = \frac{\mathrm{k}+1}{2} + 20$$

$$\Rightarrow P(k+1) \text{ is true.}$$

8.3 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x+y)$. Now,

$$(x+y)^1 = x + y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = (x+y)(x+y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is x in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x+y)^4$, the number of terms is 5.
2. The exponent of x in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.

3. The exponent of y in the first term is zero (as $y^0 = 1$). The exponent of y in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.

4. The sum of the exponents of x and y in each term is equal to the exponent of the binomial. For example, in the expansion of $(x + y)^5$, the sum of the exponents of x and y in each term is 5.

If we use the combinatorial co-efficients, we can write the expansion as

$$(x + y)^3 = {}^3C_0\, x^3 + {}^3C_1 x^2 y + {}^3C_2 x\, y^2 + {}^3C_3\, y^3$$

$$(x + y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 + {}^4C_3 x y^3 + {}^4C_4 y^4$$

$$(x + y)^5 = {}^5C_0 x^5 + {}^5C_1 x^4 y + {}^5C_2 x^3 y^2 + {}^5C_3 x^2 y^3 + {}^5C_4 x y^4 + {}^5C_5 y^5,$$

and so on.

More generally, we can write the binomial expansion of $(x + y)^n$, where n is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral**) **exponent**.

Theorem 8.1

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n \ldots(A)$$

where $n \in N$ and $x, y \in R$.

Proof : Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by $P(n)$, i.e.,

$$P(n):\ (x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \ldots + {}^nC_{n-1}\, x y^{n-1} + {}^nC_n y^n \quad \ldots(i)$$

Let us examine whether $P(1)$ is true or not.

From (*i*), we have

$$P(1) : (x + y)^1 = {}^1C_0 x + {}^1C_1\, y = 1 \times x + 1 \times y$$

i.e., $(x + y)^1 = x + y$

Thus, $P(1)$ holds.

Now, let us assume that $P(k)$ is true, i.e.,

$$P(k):\ (x + y)^k = {}^kC_0 x^k + {}^kC_1 x^{k-1} y + {}^kC_2 x^{k-2} y^2 + {}^kC_3 x^{k-3} y^3 + \ldots + {}^kC_{k-1} x y^{k-1} + {}^kC_k\, y^k \quad \ldots(ii)$$

Assuming that $P(k)$ is true, if we prove that $P(k+1)$ is true, then $P(n)$ holds, for all n. Now,

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)({}^kC_0 x^k + {}^kC_1 x^{k-1} y + {}^kC_2 x^{k-2} y^2 + \ldots + {}^kC_{k-1} x y^{k-1} + {}^kC_k y^k)$$

$$= {}^kC_0x^{k+1} + {}^kC_0x^ky + {}^kC_1x^ky + {}^kC_1x^{k-1}y^2 + {}^kC_2x^{k-1}y^2 + {}^kC_2x^{k-2}y^3 +$$

$$\dots + {}^kC_{k-1}x^2y^{k-1} + {}^kC_{k-1}xy^k + {}^kC_kxy^k + {}^kC_ky^{k+1}$$

i.e. $(x+y)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_0 + {}^kC_1)x^ky + ({}^kC_1 + {}^kC_2)x^{k-1}y^2 +$

$$\dots + ({}^kC_{k-1} + {}^kC_k)xy^k + {}^kC_ky^{k+1} \qquad \dots(iii)$$

From Lesson 7, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$ ....(iv)

and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$

Also, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore, ${}^kC_0 + {}^kC_1 = {}^{k+1}C_1$ .....(v)

$${}^kC_1 + {}^kC_2 = {}^{k+1}C_2$$

$${}^kC_2 + {}^kC_3 = {}^{k+1}C_3$$

....

.... and so on

Using (iv) and (v), we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0x^{k+1} + {}^{k+1}C_1x^ky + {}^{k+1}C_2x^{k-1}y^2 +$$

$$\dots + {}^{k+1}C_kxy^k + {}^{k+1}C_{k+1}y^{k+1}$$

which shows that P(k+1) is true.

Thus, we have shown that (a) $P(1)$ is true, and (b) if $P(k)$ is true, then $P(k+1)$ is also true.

Therefore, by the principle of mathematical induction, $P(n)$ holds for any value of n. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

Example 8.10 Write the binomial expansion of $(x + 3y)^5$.

Solution : Here the first term in the binomial is x and the second term is $3y$. Using the binomial theorem, we have

$$(x + 3y)^5 = {}^5C_0x^5 + {}^5C_1x^4(3y)^1 + {}^5C_2x^3(3y)^2 + {}^5C_3x^2(3y)^3 + {}^5C_4x(3y)^4 + {}^5C_5(3y)^5$$

$$= 1\times x^5+5x^4\times 3y+10x^3\times(9y^2)+10x^2\times(27y^3)+5x\times(81y^4)+1\times 243y^5$$

$$= x^5+15x^4y+90x^3y^2+270x^2y^3+405xy^4+243y^5$$

Thus, $(x+3y)^5 = x^5+15x^4y+90x^3y^2+270x^2y^3+405xy^4+243y^5$

Example 8.11 Expand $(1+a)^n$ in terms of powers of a, where a is a real number.

Solution : Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1+a)^n = {}^nC_0(1)^n + {}^nC_1(1)^{n-1}a + {}^nC_2(1)^{n-2}a^2 + \ldots + {}^nC_{n-1}(1)a^{n-1} + {}^nC_na^n$$

i.e., $$(1+a)^n = 1 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_{n-1}a^{n-1} + {}^nC_na^n \quad \ldots(B)$$

(B) is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x+\frac{1}{x}\right)^5, \left(\frac{y}{x}+\frac{1}{y}\right)^5, \left(\frac{a}{4}+\frac{2}{a}\right)^5, \left(\frac{2t}{3}-\frac{3}{2t}\right)^6, \text{ etc.}$$

Let us illustrate it through an example.

Example 8.12 Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, where $x, y \neq 0$.

Solution : We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2$$

$$+ {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}$$

Example 8.13 The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years ? Give the answer up to 2 decimal places.

Solution : Suppose the population is a at present. After 1 year it will be

$$a+\frac{3}{100}a=a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$=a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right)=a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places, we get

$$a\left(1+\frac{3}{100}\right)^5\approx a\left[1+5(0.03)+10(0.03)^2\right]=a\times 1.159$$

So, the increase is $0.159\times 100\%=\frac{159}{1000}\times 100\times\frac{1}{100}=15.9\%$ in 5 years.

Example 8.14 Using binomial theorem, evaluate

(*i*) 102^4 (*ii*) 97^3

Solution :

(*i*) $102^4 = (100+2)^4$

$$={}^4C_0\,(100)^4+{}^4C_1(100)^3.2+{}^4C_2(100)^2.2^2+{}^4C_3\,(100).2^3+{}^4C_4\,.2^4$$

$$=100000000+8000000+240000+3200+16$$

$$=108243216$$

(*ii*) $(97)^3=(100-3)^3$

$$={}^3C_0(100)^3-{}^3C_1(100)^2.3+{}^3C_2(100).3^2-{}^3C_3.3^3$$

$$=1000000-90000+2700-27$$

$$=1002700-90027$$

$$=912673$$

CHECK YOUR PROGRESS 8.3

1. Write the expansion of each of the following :

   (a) $(2a+b)^3$ (b) $(x^2-3y)^6$ (c) $(4a-5b)^4$ (d) $(ax+by)^n$

2. Write the expansions of :

   (a) $(1-x)^7$ (b) $\left(1+\frac{x}{y}\right)^7$ (c) $(1+2x)^5$

3. Write the expansions of :

   (a) $\left(\frac{a}{3}+\frac{b}{2}\right)^5$ (b) $\left(3x-\frac{5}{x^2}\right)^7$ (c) $\left(x+\frac{1}{x}\right)^4$ (d) $\left(\frac{x}{y}+\frac{y}{x}\right)^5$

4. Suppose I invest Rs. 1 lakh at 18% per year compound interest. What sum will I get back after 10 years? Give your answer up to 2 decimal places.

5. The population of bacteria increases at the rate of 2% per hour. If the count of bacteria at 9 a.m. is 1.5×10^5, find the number at 1 p.m. on the same day.

6. Using binomial theorem, evaluate each of the following :

   (*i*) $(101)^4$ (*ii*) $(99)^4$ (*iii*) $(1.02)^3$ (iv) $(0.98)^3$

8.4 GENERAL AND MIDDLE TERMS IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x+y)^n$, i.e., in

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

We observe that

the first term is ${}^nC_0x^n$, i.e., ${}^nC_{1-1}x^ny^0$;

the second term is ${}^nC_1x^{n-1}y$, i.e., ${}^nC_{2-1}x^{n-1}y^1$;

the third term is ${}^nC_2x^{n-2}y^2$, i.e., ${}^nC_{3-1}x^{n-2}y^2$;

and so on.

From the above, we can generalise that

the $(r+1)^{th}$ term is ${}^nC_{(r+1)-1}x^{n-r}y^r$, i.e., ${}^nC_rx^{n-r}y^r$.

If we denote this term by T_{r+1}, we have

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

T_{r+1} is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

Example 8.15 Find the $(r+1)^{th}$ term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$, where n is a natural number. Verify your answer for the first term of the expansion.

Solution : The general term of the expansion is given by :

$$T_{r+1} = {}^nC_r (x^2)^{(n-r)} \left(\frac{1}{x}\right)^r$$

$$= {}^nC_r x^{2n-2r} \frac{1}{x^r}$$

$$= {}^nC_r x^{2n-3r} \qquad \text{...}(i)$$

Hence, the $(r+1)$th term in the expansion is ${}^nC_r x^{2n-3r}$.

On expanding $\left(x^2 + \frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or x^{2n}.

Using (*i*), we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$

$$\therefore \qquad T_1 = {}^nC_0 x^{2n-0} = x^{2n}$$

This verifies that the expression for T_{r+1} is correct for $r + 1 = 1$.

Example 8.16 Find the fifth term in the expansion of

$$\left(1 - \frac{2}{3}x^3\right)^6$$

Solution : Using here $T_{r+1} = T_5$ which gives $r + 1 = 5$, i.e., $r = 4$.

Also $n = 6$ and let $a = \frac{-2}{3}x^3$.

$$T_5 = {}^6C_4 \left(-\frac{2}{3}x^3\right)^4$$

$$= {}^6C_2\left(\frac{16}{81}x^{12}\right)$$

$$= \frac{6\times 5}{2}\times\frac{16}{81}\times x^{12} = \frac{80}{27}x^{12}$$

Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$

CHECK YOUR PROGRESS 8.4

1. For a natural number *n*, write the $(r+1)^{\text{th}}$ term in the expansion of each of the following:

(a) $(2x+y)^n$ (b) $(2a^2-1)^n$

(c) $(1-a)^n$ (d) $\left(3+\frac{1}{x^2}\right)^n$

2. Find the specified terms in each of the following expansions:

(a) $(1+2y)^8$; 6th term (b) $(2x+3)^7$; 4th term

(c) $(2a-b)^{11}$; 7th term (d) $\left(x+\frac{1}{x}\right)^6$; 4th term

(e) $\left(x^3-\frac{1}{x^2}\right)^7$; 5th term

Now that you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even. Thus, while finding the middle term in a binomial expansion, we come across two cases:

Case 1 : When *n* is even.

To study such a situation, let us look at a particular value of *n*, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 8.1, you can see that there are three terms on either side of the fourth term.

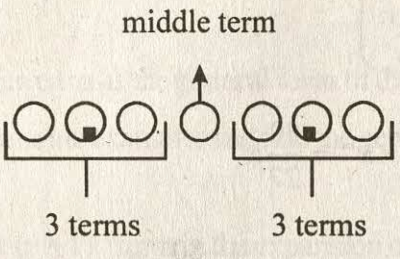


Fig. 8.1

In general, when the exponent *n* of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

Case 2: When *n* is odd

Let us take $n = 7$ as an example to see what happens in this case. The number of terms in the expansion will be 8. Looking at Fig. 8.2, do you find any one middle term in it? There is not. But we can partition the terms into two equal parts by a line as shown in the figure. We call the terms on either side of the partitioning line taken together, the middle terms. This is because there are an equal number of terms on either side of the two, taken together.

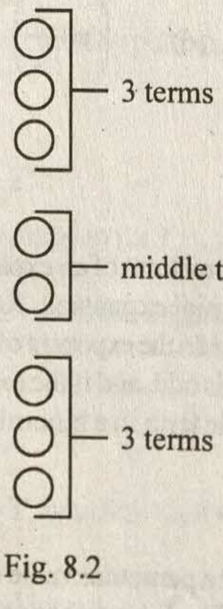


Fig. 8.2

Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if $n = 13$, then the $\left(\frac{13+1}{2}\right)$th and the $\left(\frac{13+3}{2}\right)$th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 8.3.

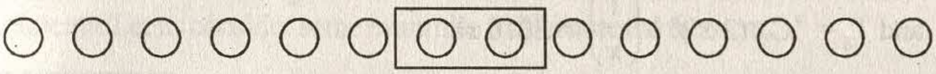


Fig. 8.3

From the above, we conclude that

When the exponent n of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are two middle terms in the corresponding binomial expansion.

Let us now consider some examples.

Example 8.17 Find the middle term in the expansion of $(x^2+y^2)^8$.

Solutuion : Here $n = 8$ (an even number).

Therefore, the $\left(\frac{8}{2}+1\right)$th, i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r(x^2)^{8-r}y^r$,

$$T_5 = {}^8C_4(x^2)^{8-4}(y^2)^4 = 70x^8y^8$$

Example 8.18 Find the middle term(s) in the expansion of $\left(2x^2+\frac{1}{x}\right)^9$.

Solution : Here $n = 9$ (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)th$ and $\left(\frac{9+3}{2}\right)th$ are middle terms. i.e. T_5 and T_6 are middle terms.

For finding T_5 and T_6, putting $r = 4$ and $r = 5$ in the general term

$$T_{r+1} = {}^9C_r(2x^2)^{9-r}\left(\frac{1}{x}\right)^r,$$

$$T_5 = {}^9C_4(2x^2)^{9-4}\left(\frac{1}{x}\right)^4$$

$$= \frac{9\times8\times7\times6}{4\times3\times2}\times(32x^{10})\times\left(\frac{1}{x}\right)^4 = 4032\,x^6$$

$$\text{and } T_6 = {}^9C_5\,(2\,x^2)^{9-5}\left(\frac{1}{x}\right)^5 = 2016\,x^3$$

Thus, the two middle terms are $4032\,x^6$ and $2016\,x^3$.

CHECK YOUR PROGRESS 8.5

1. Find the middle term(s) in the expansion of each of the following :

   (a) $(2x+y)^{10}$ (b) $\left(1+\frac{2}{3}x^3\right)^8$

   (c) $\left(x+\frac{1}{x}\right)^6$ (d) $\left(1-x^2\right)^{10}$

2. Find the middle term(s) in the expansion of each of the following :

   (a) $(a+b)^7$ (b) $(2a-b)^9$

   (c) $\left(\frac{3x}{4}-\frac{4y}{3}\right)^7$ (d) $\left(x+\frac{1}{x^2}\right)^{11}$

8.5 BINOMIAL THEOREM FOR RATIONAL EXPONENTS

So far you have applied the binomial theorem only when the binomial has been raised to a power which is a natural number. What happens if the exponent is a negative integer, or if it is a fraction? We will state the result that allows us to still have a binomial expansion, but it will have infinite terms in this case.

The result is a generalised version of the earlier binomial theorem which you have studied.

Theorem 8.2 **The Binomial Theorem for a Rational Exponent.**

If r is a rational number, and x is a real number such that $|x| < 1$, *then*

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots \qquad \text{... (D)}$$

We will not prove this result here, as it is beyond the scope of this course. In fact, even Sir Issac Newton, who is credited with stating this generalisation, stated it without proof in two letters, written in A.D. 1676. The proof was developed later, by other mathematicians, in stages. Among those who contributed to the proof of this theorem were English mathematician Colin Maclaurin (A.D. 1698-1746) for rational values of r, Giovanni Francesco, M.M. Salvemini (A.D. 1708-1783) and the German mathematician Abraham G. Kasther (A.D. 1719 - 1800) for integral values of r, the Swiss mathematician Leonhard Euler (AD 1707-1783) for fractional exponents and the Norwegian mathematician Neils Henrik Abel (1802-1829) for complex exponents. Let us consider some examples to illustrate the theorem.

Example 8.19 Write the expansion of $(1+x)^{-1}$, when $|x| < 1$.

Solution : Here $r = -1$ [with reference to (D) above].

Therefore,

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots$$

i.e., $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots.$

Similarly, you can write the expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots.$

Note the above expansions. In case of $(1+x)^{-1}$ all the terms have positive and negative signs alternate, while in the case of $(1-x)^{-1}$ all the terms have positive sign.

You may have also observed the following points about the binomial expansion (D) in general;

1. If r is a natural number, then (C) and (D) coincide for the case $|x| < 1$.
2. Note that ${}^rC_0 = 1$, ${}^rC_1 = r$, ${}^rC_2 = \frac{r(r-1)}{2!}$ etc. Thus, the coefficients $1, r, \frac{r(r-1)}{2!} \ldots$ in (D) look like combinatorial coefficients.,

   However, recall that ${}^rC_s$ is defined for natural numbers r and whole number s only. Therefore,

   ${}^rC_0, {}^rC_1, {}^rC_2,$ etc. have no meaning in the present context.
3. The expression (D) will have an infinite number of terms.
4. The sum of the series on the RHS of (D) may not be meaningful if $x > 1$.

   For example, if we put $x = 2$ in Example 1, we have

   $(1+2)^{-1} = 1 - 2 + 4 - 8 + 16 - 32 + \ldots$

   i.e., $\frac{1}{3} = (1-2) + (4-8) + (16-32) + \ldots$

   i.e., $\frac{1}{3} = -1 - 4 - 16 - \ldots,$

which is clearly false.

Therefore, for (D) to hold, it is necessary that $|x| < 1$.

Let us look at some more examples of this binomial expansion.

Example 8.20 Expand $(x+y)^r$, where r is a rational number and $\left|\frac{y}{x}\right| < 1$.

Hence expand $(3+5p)^{2/5}$, when $|p| < \frac{3}{5}$.

Solution : $(x+y)^r = x^r\left(1+\frac{y}{x}\right)^r$ .... (1)

Since it is given that $\left|\frac{y}{x}\right| < 1$, we have

$$\left(1+\frac{y}{x}\right)^r = 1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \dots$$

Therefore, from (1), we have

$$(x+y)^r = x^r\left[1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \dots\right]$$

$$\text{i.e., } (x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \dots \quad (2)$$

Now, to solve the second part of the question, note that $|p| < \frac{3}{5}$.

If $\left|\frac{5p}{3}\right| < 1$, then putting $x = 3,\ y = 5p,\ r = \frac{2}{5}$ in (2), we get

$$(3+5p)^{2/5} = 3^{2/5} + \frac{2}{5}(3)^{\frac{2}{5}-1}(5p)^1 + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)}{2!}(3)^{\frac{2}{5}-2}(5p)^2 + \dots$$

$$= 3^{2/5} + (3)^{-3/5}(2p) + \frac{\frac{2}{5}\left(-\frac{3}{5}\right)}{2}(3)^{-8/5}25p^2 + \ldots$$

$$= 3^{2/5} + 3^{-3/5}(2p) - 3^{-3/5}p^2 + \ldots$$

The result we have just obtained in Example 8.20 is another form of the binomial theorem for a **rational exponent.** Let us restate it formally.

If r is a rational number and $\left|\frac{y}{x}\right| < 1$,

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \ldots\ldots \text{ (E)}$$

Note that you could have expanded $(x+y)^r$ differently if $\left|\frac{y}{x}\right| > 1$ were true. In this case, you would have had $\left|\frac{x}{y}\right| < 1$, and $(x+y)^r = y^r\left(1+\frac{x}{y}\right)^r = y^r + ry^{r-1}.x + \ldots$

Consequently, we have the following result:

For a rational number r, an expression like $(ax+by)^r$ can be expanded in two different ways, depending on whether

$$\left|\frac{by}{ax}\right| < 1 \text{ or } \left|\frac{ax}{by}\right| < 1$$

Example 8.21 Expand $(x+y)^{-5}$ when

(i) $\left|\frac{y}{x}\right| < 1$ and (ii) $\left|\frac{x}{y}\right| < 1$

Solution : (i) Since $\left|\frac{y}{x}\right| < 1$, using (E) we have

$$(x+y)^{-5} = x^{-5} + (-5)x^{-5-1}y + \frac{(-5)(-6)}{2!}x^{-5-2}y^2 + \frac{(-5)(-6)(-7)}{3!}x^{-5-3}y^3 + \ldots.$$

$$= \frac{1}{x^5} - \frac{5y}{x^6} + \frac{15y^2}{x^7} - \frac{35y^3}{x^8} + \ldots$$

(ii) Since $\left|\frac{x}{y}\right| < 1$, we have to write $(x+y)^{-5}$ in the form $(y+x)^{-5}$.

Using (E), we can write

$$(y+x)^{-5} = y^{-5} + (-5)\,y^{-5-1}x$$

$$+\frac{(-5)(-6)}{2!}\,y^{-5-2}x^{2} + \frac{(-5)(-6)(-7)}{3!}\,y^{-5-3}x^{3} + \ldots$$

$$= \frac{1}{y^{5}} - \frac{5x}{y^{6}} + \frac{15x^{2}}{y^{7}} - \frac{35x^{3}}{y^{8}} + \ldots.$$

Note that in (i), we have obtained the expansion in ascending powers of y while in (ii), we have obtained the expansion in ascending powers of x.

CHECK YOUR PROGRESS 8.6

1. Expand each of the following :

   (a) $(1-p)^{-3}$ for $|p|<1$　　(b) $(1+3x)^{4/3}$ for $|x|<\frac{1}{3}$

   (c) $(1-5z)^{\frac{6}{5}}$, for $|z|<\frac{1}{5}$

2. Expand each of the following :

   (a) $(27-6x)^{\frac{-2}{3}}$, for $\left|\frac{2x}{9}\right|<1$　　(b) $(2a+x)^{-3}$, for $\left|\frac{x}{2a}\right|<1$

   (c) $(2+3y)^{\frac{1}{7}}$, for $|y|>\frac{2}{3}$

3. (a) State the condition under which the expansion of $(x+2y)^{-5}$ will be valid in

   (*i*) ascending powers of x.

   (*ii*) ascending powers of y.

   Also, write down the expansion in each case.

   (b) Expand, $(3+6y)^{\frac{-4}{3}}$, stating the range of values of y for which the expansion is valid.

8.6 USE OF BINOMIAL THEOREM IN APPROXIMATIONS

As you have seen, the binomial expansions sometime have infinitely many terms. In such cases, for further calculations; an approximate value involving only the first few terms may be enough for us. Let us illustrate some situations in which we find the approximate values.

Example 8.22 Find the cube root of 1.03 up to three decimal places.

Solution : We want to find $(1.03)^{\frac{1}{3}}$ up to three decimal places.

Now $(1.03)^{\frac{1}{3}} = (1+0.03)^{\frac{1}{3}}$

Since $|0.03| < 1$, from (E), we have

$$(1+0.03)^{\frac{1}{3}} = 1+\frac{1}{3}(0.03)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.03)^2+\ldots \qquad (i)$$

Now, we need to approximate the value up to three decimal places. Since a non-zero digit in the fourth decimal place may affect the digit in the third place in the process of rounding off, we need to consider those terms in the expansion which produce a non-zero digit in the first, second, third or fourth decimal place.

Therefore, we can take the sum of the first three terms in the Expansion (*i*), and ignore the rest.

$$\therefore \quad (1.03)^{\frac{1}{3}} \approx 1+0.01+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(0.0009)$$

$$= 1+0.01-0.0001$$

$$= 1.0099$$

$\approx$ 1.010, taking the value up to three decimal places.

Now, the digit after the third decimal place is greater than 5, so we have increased the third decimal place by 1.

Thus, the cube root of 1.03, up to three decimal places, is 1.010.

Example 8.23 Assuming y to be so small that y^2 and higher powers of y can be neglected, find the value of $(1-2y)^{\frac{2}{3}}(4+5y)^{-\frac{3}{2}}$

Solution : Note that y is very small. So, we can assume that $|y| < \frac{1}{2}$. Then, using the binomial theorem, we get

$$(1-2y)^{\frac{2}{3}} = 1+\frac{2}{3}(-2y)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}(-2y)^2+...$$

and

$$(4+5y)^{\frac{-3}{2}} = 4^{\frac{-3}{2}}+\left(-\frac{3}{2}\right)(4)^{\frac{-3}{2}-1}(5y)+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}(4)^{\frac{-3}{2}-2}.(5y)^2+...$$

Since we can neglect terms containing y^2 and higher powers of y, we have

$$(1-2y)^{\frac{2}{3}} \approx 1+\frac{2}{3}(-2y)=1-\frac{4}{3}y, \text{ and}$$

$$(4+5y)^{\frac{-3}{2}} \approx (4)^{\frac{-3}{2}}-\frac{3}{2}(4)^{\frac{-5}{2}}(5y)$$

$$= \frac{1}{8}-\frac{15}{64}y$$

Thus, the given product is approximately

$$\left(1-\frac{4}{3}y\right)\left(\frac{1}{8}-\frac{15}{64}y\right)=\frac{1}{8}-\frac{1}{6}y-\frac{15}{64}y+\frac{5}{16}y^2$$

$$\approx \frac{1}{8}-\frac{77}{192}y, \text{ again neglecting the term containing } y^2.$$

So, $(1-2y)^{\frac{2}{3}}\ (4+5y)^{\frac{-3}{2}}$ is $\frac{1}{8}-\frac{77}{192}y$, if we neglect the terms involving y^2 and higher powers of y.

CHECK YOUR PROGRESS 8.7

1. Find the value of each of the following up to three decimal places:

(a) $(1.02)^2$ (b) $(1.01)^{-3}$ (c) $(0.97)^{-4}$

(d) $\sqrt[3]{7.60}$ [Hint : $(7.60)^{1/3} = (8-0.4)^{1/3}$]

(e) $\sqrt[4]{82}$ [Hint: $(82)^{\frac{1}{4}}=(81+1)^{\frac{1}{4}}=3\left(1+\frac{1}{81}\right)^{\frac{1}{4}}$]

(f) $(24)^{\frac{-1}{2}}$ [Hint : $(24)^{\frac{-1}{2}}=(25-1)^{\frac{-1}{2}}$]

2. Assuming z to be so small that z^2 and higher powers of z can be neglected, find the value of

(a) $(3+2z)^{-5}$ (b) $(1+3z)^{\frac{2}{3}}(1-5z)^{-2}$

(c) $\dfrac{\sqrt{1+z}+(1-z)^{2/3}}{(1+z)+\sqrt{1+z}}$ [Hint : LHS $\approx \dfrac{1+\frac{1}{2}z+1-\frac{2}{3}z}{(1+z)+(1+\frac{1}{2}z)}$]

(d) $\dfrac{(1-z)^{\frac{1}{3}}+(11-5z)^2}{\sqrt[4]{16-z}}$

LET US SUM UP

- The statement of the principle of mathematical induction namely.

  P(n), a statement involving a natural number n, is true for all $n \geq 1$, where n is a fixed natural number, if

  (i) P (1) is true, and

  (ii) Whenever P(k) is true, then P ($k+1$) is true for $k \in N$.

- For a natural number n,

  $$(x+y)^n = {}^nC_o x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n$$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1+a)^n = {}^nC_o + {}^nC_1 a + {}^nC_2 a^2 + \ldots. + {}^nC_{n-1} a^{n-1} + {}^nC_n a^n$

- The general term in the expansion of $(x+y)^n$ is ${}^nC_r x^{n-r} y^r$ and in the expansion of $(1+a)^n$ is ${}^nC_r a^r$, where n is a natural number and $0 \leq r \leq n$.

- If n is an even natural number, there is only one middle term in the expansion of $(x+y)^n$. If n is odd, there are two middle trems in the expansion.

- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.

- The statement

  $$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \ldots.$$

  where, r is a rational number and $|x| < 1$ is called the **Binomial Theorm for a rational exponent.** In this expansion, the number of terms is infinite if r is not a whole number.

- $(x+y)^r = x^r + r x^{r-1} y + r\frac{(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \ldots.,$

where r is a rational number and $\left|\frac{y}{x}\right| < 1$, is another form of the Binomial Theorem for a rational exponent.

- Expressions like $(ax+by)^r$, where r is a rational number, can be expanded in two different ways, depending on whether $\left|\frac{by}{ax}\right| < 1$ or $\left|\frac{ax}{by}\right| < 1$.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Verify each of the following statements, using the principle of mathematical induction :

   (a) The number of subsets of a set with n elements is 2^n.

   (b) $(a+b)^n > a^n + b^n \ \forall n \geq 2$, where a and b are positive real numbers.

   (c) $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n-1)}{r-1}$, where $r > 1$ and a is a real number.

   (d) $(x^{2n}-1)$ is divisible by $(x+1) \ \forall x \in N$.

   (e) $(10^{2n-1}+1)$ is a multiple of 11.

   [Hint: $10^{2k+1} + 1 = 10^2[(10^{2k-1}+1) - 99]$**]**

   (f) $(4.10^{2n} + 9.10^{2n-1} + 5)$ is a multiple of 99.

   (g) $n(n^2-1)$ is a multiple of 24, when n is odd.

   [Since n is odd, assume that $P(2k+1)$ is true, as $(2k+3)$ is always odd. Then try to prove that $P(2k+3)$ is true.]

   (h) $(1+x)^n > 1 + nx$, where $x > 0$.

   (i) If f and g are polynomials in x with real coefficients and $f+g \neq 0$, then $(f+g)$ divides $(f^{2n-1} + g^{2n-1}) \ \forall n \in N$.

2. Write the expansion of each of the following :

   (a) $(3x+2y)^5$ (b) $(p-q)^8$ (c) $(1-x)^8$

(d) $\left(1+\frac{2}{3}x\right)^6$ (e) $\left(x+\frac{1}{2x}\right)^6$ (f) $\left(3x-y^2\right)^5$

(g) $\left(\frac{x^2}{4}+\frac{2}{x}\right)^4$ (h) $\left(x^2-\frac{1}{x^3}\right)^7$ (i) $\left(x^3+\frac{1}{x^2}\right)^5$

(j) $\left(\frac{1}{x^2}-x^3\right)^4$

3. Write the (r + 1)th term in the expansion of each of the following, where $n \in N$:

(a) $(3x-y^2)^n$ (b) $\left(x^3+\frac{1}{x}\right)^n$

4. Find the specified terms in the expansion of each of the following :

(a) $(1-2x)^7$: 3rd term [Hint : Here $r = 2$] (b) $\left(x+\frac{1}{2x}\right)^6$: middle term (s)

(c) $(3x-4y)^6$: 4th term (d) $\left(y^2-\frac{1}{y}\right)^{11}$: middle term (s)

(e) $(x^3-y^3)^{12}$: 4th term (f) $(1-3x^2)^{10}$: middle term (s)

(g) $(-3x-4y)^6$: 5th term (h) Write the rth term in the expansion of $(x-2y)^6$.

(i) Write the $(r-1)$ th term in the expansion of $(1+2x)^8$.

5. If T_r, denotes the rth term in the expansion of $(1+x)^n$ in ascending powers of x (n being a natural number), prove that

$r(r+1)T_{r+2} = (n-r+1)(n-r)x^2 T_r$

[Hint : $T_r = {}^nC_{r-1}\,x^{r-1}$ and $T_{r+2} = {}^nC_{r+1}\,x^{r+1}$**]**

6. k_r is the coefficient of x^{r-1} in the expansion of $(1+2x)^{10}$ in ascending powers of x and $k_{r+2} = 4k_r$. Find the value of r.

[Hint : $k_r = {}^{10}C_{r-1}\,2^{r-1}$ and $k_{r+2} = {}^{10}C_{r+1}2^{r+1}$**]**

7. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1+a)^n$ (n being a natural number) are in A.P. Find n.

[Hint : ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$**]**

8. Expand $(1+y+y^2)^4$. **[Hint :** $\left(1+y+y^2\right)^4 = \left\{(1+y)+y^2\right\}^4$**]**

9. Write the expansion of each of the following :

(a) $(1-x)^{-4}, |x|<1$ (b) $\frac{1}{(1+x)^3}, |x|<1$

(c) $(3-z)^{-4}, |z|<3$ (d) $\frac{1}{(1+3x)^{3/2}}, |x|<\frac{1}{3}$

10. State the condition under which the expansion of $(x-2y)^{-3}$ will be valid in ascending powers of y. Also write the expansion.

11. State the condition under which the expansion of $(x-3y)^{\frac{-1}{2}}$ will be valid in ascending powers of x. Also write the expansion.

12. Expand the following, stating the condition of y under which the expansion will be valid :

(a) $\frac{1}{(2+y)^4}$ (b) $(3-y)^{\frac{-2}{3}}$

13. Find the value of each of the following up to three decimal places, using the necessary number of terms in the expansion:

(a) $(0.99)^{-4}$ (b) $(1.03)^{-3}$

(c) $\sqrt[3]{26}$ **[Hint :** $(26)^{\frac{1}{3}}=(27-1)^{\frac{1}{3}}$]

(d) $\sqrt[7]{127}$ **[Hint :** $(127)^{\frac{1}{7}}=(128-1)^{\frac{1}{7}}$]

(e) $\sqrt[5]{35}$ **[Hint :** $(35)^{\frac{1}{5}}=\{32+3\}^{\frac{1}{5}}$]

(f) $\sqrt[5]{31}$ **[Hint :** $(31)^{\frac{1}{5}}=\{32-1\}^{\frac{1}{5}}$]

(g) $\sqrt[3]{1001}$ **[Hint :** $(1001)^{\frac{1}{3}}=(1000+1)^{\frac{1}{3}}$]

14. Assuming y to be so small that y^2 and higher powers of y can be neglected, find the value of each of the following :

(a) $(1+5y)^{-2}(1+2y)^{\frac{-3}{2}}$ (b) $\frac{(1-4y)^{-3}(1-2y^2)^{\frac{1}{2}}}{(4-y)^{3/2}}$

(c) $\frac{\sqrt{1-3y}+(1-y)^{\frac{5}{3}}}{\sqrt{4-y}}$

15. A student puts $n = 0$ in $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + x^n$ and obtains $(1+x)^0 = 1 + 0 + 0 + \ldots + x^0$, i.e. $1 = 1 + 1 = 2$. Can you detect the error in this solution?

16. Assuming that the expansions are possible, find the coefficient of y^3 in $(1-4y)^2(1-2y^2)^{1/2}$.

17. Prove that $(1+x+x^2+x^3+\ldots)(1-x+x^2-x^3+\ldots) = 1+x^2+x^4+x^6+\ldots$

[**Hint :** LHS $= (1-x)^{-1}(1+x)^{-1} = (1-x^2)^{-1}$]

ANSWERS

CHECK YOUR PROGRESS 8.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of *n*, and therefore we are not in a position to decide, if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

   Note that (*f*) is universally false.

2. P(1) : 6 is a factor of $1^3 + 5.1$

   P(2) : 6 is a factor of $2^3 + 5.2$

   P(*k*) : 6 is a factor of $k^3 + 5k$

   P(*k*+1) : 6 is a factor of $(k+1)^3 + 5(k+1)$

3. (a) $P(1) : 2 \geq 2$

   $P(k) : 2^k \geq k + 1$

   $P(k+1) : 2^{k+1} \geq k + 2$

   (b) $P(1) : 1 + x \geq 1 + x$

   $P(k) : (1+x)^k \geq 1 + kx$

   $P(k+1) : (1+x)^{k+1} \geq 1 + (k+1)x$

   (c) $P(1)$: 6 is divisible by 6.

   $P(k) : k(k+1)(k+2)$ is divisible by 6.

   $P(k+1) : (k+1)(k+2)(k+3)$ is divisible by 6

   (d) $P(1)$: $(x - y)$ is divisible by $(x - y)$.

   $P(k)$: $(x^k - y^k)$ is divisible by $(x - y)$

   $P(k+1)$: $(x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) $P(1)$: $ab = ab$

   $P(k)$: $(ab)^k = a^k b^k$

   P(*k*+1): $(ab)^{k+1} = a^{k+1} . b^{k+1}$

(f) $P(1): \frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ is a natural number.

$P(k)$: $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

$P(k+1)$: $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is a natural number.

4. (a) $P(1)$: $\frac{1}{1\times 2}=\frac{1}{2}$

$P(2): \frac{1}{1\times 2}+\frac{1}{2\times 3}=\frac{2}{3}$

$P(k): \frac{1}{1\times 2}+\frac{1}{2\times 3}+...+\frac{1}{k(k+1)}=\frac{k}{k+1}$

$P(k+1): \frac{1}{1\times 2}+...+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

(b) $P(1)$: $1 = 1^2$

$P(2)$: $1+3=2^2$

$P(k)$: $1+3+5+...+(2k-1)=k^2$

$P(k+1)$: $1+3+5+...+(2k-1)+[2(k+1)-1]=(k+1)^2$

(c) $P(1): 1\times 2<1(2)^2$

$P(2): (1\times 2)+(2\times 3)<2(3)^2$

$P(k): (1\times 2)+(2\times 3)+...+k(k+1)<k(k+1)^2$.

$P(k+1): (1\times 2)+(2\times 3)+...+(k+1)(k+2)<(k+1)(k+2)^2$

(d) $P(1)$: $\frac{1}{1\times 3}=\frac{1}{3}$

$P(2)$: $\frac{1}{1\times 3}+\frac{1}{3\times 5}=\frac{2}{5}$

$P(k)$: $\frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

$P(k+1): \frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$

CHECK YOUR PROGRESS 8.3

1. (a) $8a^3+12a^2b+6ab^2+b^3$

   (b) $x^{12}-18x^{10}y+135x^8y^2-540x^6y^3+1215x^4y^4-1458x^2y^5+729y^6$

   (c) $256a^4-1280a^3b+2400a^2b^2-2000ab^3+625b^4$

   (d) $a^nx^n+na^{n-1}x^{n-1}by+\dfrac{n(n-1)}{2!}a^{n-2}x^{n-2}b^2y^2+\ldots+b^ny^n$

2. (a) $1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7$

   (b) $1+\dfrac{7x}{y}+\dfrac{21x^2}{y^2}+\dfrac{35x^3}{y^3}+\dfrac{35x^4}{y^4}+\dfrac{21x^5}{y^5}+\dfrac{7x^6}{y^6}+\dfrac{x^7}{y^7}$

   (c) $1+10x+40x^2+80x^3+80x^4+32x^5$

3. (a) $\dfrac{a^5}{243}+\dfrac{5a^4b}{162}+\dfrac{5a^3b^2}{54}+\dfrac{5a^2b^3}{36}+\dfrac{5ab^4}{48}+\dfrac{b^5}{32}$

   (b) $2187x^7-25515x^4+127575x-\dfrac{354375}{x^2}+\dfrac{590625}{x^5}-\dfrac{590625}{x^8}$

   $+\dfrac{328125}{x^{11}}-\dfrac{78125}{x^{14}}$

   (c) $x^4+4x^2+6+\dfrac{4}{x^2}+\dfrac{1}{x^4}$

   (d) $\dfrac{x^5}{y^5}+5\dfrac{x^3}{y^3}+10\dfrac{x}{y}+10\dfrac{y}{x}+5\dfrac{y^3}{x^3}+\dfrac{y^5}{x^5}$

4. Rs 4.96 lakh

5. 162360

6. (i) 104060401

   (ii) 96059601

   (iii) 1.061208

   (iv) 0.941192

CHECK YOUR PROGRESS 8.4

1. (a) $^nC_r 2^{n-r} x^{n-r} y^r$
   (b) $^nC_r 2^{n-r} a^{2n-2r} (-1)^r$
   (c) $^nC_r (-1)^r a^r$
   (d) $^nC_r 3^{n-r} . x^{-2r}$
2. (a) $1792y^5$
   (b) $15120x^4$
   (c) $14784a^5b^6$
   (d) 20
   (e) $35x$

CHECK YOUR PROGRESS 8.5

1. (a) $8064x^5y^5$
   (b) $\frac{1120}{81}x^{12}$
   (c) 20
   (d) $-252x^{10}$
2. (a) $35a^4b^3, 35a^3b^4$
   (b) $4032a^5b^4, -2016a^4b^5$
   (c) $\frac{-105}{4}x^4y^3, \frac{140}{3}x^3y^4$
   (d) $\frac{462}{x^4}, \frac{462}{x^7}$

CHECK YOUR PROGRESS 8.6

1. (a) $1 + 3p + 6p^2 + 10p^3 + ...$
   (b) $1 + 4x + 2x^2 + ...$
   (c) $1 - 6z + 3z^2 + 4z^3 + ...$

2. (a) $\frac{1}{9}+\frac{4x}{243}+\frac{20}{6561}x^2+\ldots$

(b) $\frac{1}{8a^3}-\frac{3x}{16a^4}+\frac{3x^2}{16a^5}-\ldots$

(c) $3^{\frac{1}{7}}y^{\frac{1}{7}}\ [1+\frac{2}{21y}-\frac{4}{147y^2}+\ldots]$

3. (a) (i) $\left|\frac{x}{2y}\right|<1: \frac{1}{32y^5}-\frac{5x}{64y^6}+\frac{15x^2}{128y^7}-\ldots$

(ii) $\left|\frac{2y}{x}\right|<1: \frac{1}{x^5}-\frac{10y}{x^6}+\frac{60y^2}{x^7}-\ldots$

(b) $|y|<\frac{1}{2}: 3^{-4/3}-8(3)^{-7/3}y+56(3)^{-10/3}y^2+\ldots$

CHECK YOUR PROGRESS 8.7

1. (a) 1.041

(b) 0.971

(c) 1.130

(d) 1.968

(e) 3.009

(f) 0.204

2. (a) $\frac{1}{243}-\frac{10z}{729}$

(b) $1+12z$

(c) $1-\frac{5z}{6}$

(d) $61-\frac{10409}{192}z$

TERMINAL EXERCISE

2. (a) $243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4+32y^5$

(b) $p^8-8p^7q+28p^6q^2-56p^5q^3+70p^4q^4-56p^3q^5+28p^2q^6-8pq^7+q^8$

(c) $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$

(d) $1+4x+\frac{20}{3}x^2+\frac{160}{27}x^3+\frac{80}{27}x^4+\frac{64}{81}x^5+\frac{64}{729}x^6$

(e) $x^6+3x^4+\frac{15}{4}x^2+\frac{5}{2}+\frac{15}{16x^2}+\frac{3}{16x^4}+\frac{1}{64x^6}$

(f) $243x^5-405x^4y^2+270x^3y^4-90x^2y^6+15xy^8-y^{10}$

(g) $\frac{x^8}{256}+\frac{x^5}{8}+\frac{3}{2}x^2+\frac{8}{x}+\frac{16}{x^4}$

(h) $x^{14}-7x^9+21x^4-\frac{35}{x}+\frac{35}{x^6}-\frac{21}{x^{11}}+\frac{7}{x^{16}}-\frac{1}{x^{21}}$

(i) $x^{15}+5x^{10}+10x^5+10+\frac{5}{x^5}+\frac{1}{x^{10}}$

(j) $\frac{1}{x^8}-\frac{4}{x^3}+6x^2-4x^7+x^{12}$

3. (a) $(-1)^r\,{}^nC_r3^{n-r}x^{n-r}y^{2r}$

(b) ${}^nC_rx^{3n-4r}$

4. (a) $84x^2$

(b) $\frac{5}{2}$

(c) $-34560\,x^3y^3$

(d) $-462y^7,\ 462y^4$

(e) $-220x^{27}y^9$

(f) $-61236\,x^{10}$

(g) $34560x^2y^4$

(h) $(-2)^{r-1}\,{}^6C_{r-1}x^{7-r}y^{r-1}$

(i) $-2^{r-2}\ {}^8C_{r-2}\,x^{r-2}$

6. 5 7. 7, 14

8. $1 + 4y + 10y^2 + 16y^3 + 19y^4 + 16y^5 + 10y^6 + 4y^7 + y^8$

9. (a) $1 + 4x + 10x^2 + \ldots$

(b) $1 - 3x + 6x^2 - 10x^3 + \ldots$

(c) $\frac{1}{81} + \frac{4}{243}z + \frac{10}{729}z^2 + \ldots$

(d) $1 - \frac{9}{2}x + \frac{135}{8}x^2 - \frac{945}{16}x^3 + \ldots$

10. $\left|\frac{2y}{x}\right| < 1 : \frac{1}{x^3} + \frac{6y}{x^4} + \frac{24y^2}{x^5} + \ldots$

11. $\left|\frac{x}{3y}\right| < 1 : \frac{1}{\sqrt{-3y}} + \frac{x}{6y\sqrt{-3y}} + \frac{x^2}{24y^2\sqrt{-3y}} + \ldots$

12. (a) $\frac{1}{16} - \frac{y}{8} + \frac{5y^2}{32} - \frac{5y^3}{32} + \ldots |y| < 2$

(b) $\frac{1}{3^{2/3}} + \frac{2x}{9 \times 3^{2/3}} + \frac{5x^2}{81 \times 3^{2/3}} + \ldots, |y| < 3$

13. (a) 1.041

(b) 0.915

(c) 2.833

(d) 1.998

(e) 2.037

(f) 1.987

(g) 10.003

14. (a) $1 - 13y$

(b) $\frac{8 + 99y}{64}$

(c) $1 - \frac{35}{24}y$

15. Expansion is valid when n is a natural number. Here, $n = 0$

16. 8.

SENIOR SECONDARY COURSE

MATHEMATICS

Questions For Practice - Algebra

Maximum Marks : 50 Time : $1\frac{1}{2}$ Hours

Instructions

1. Answer all questions on a separate sheet of paper.

2. Give the following information on your answer sheet:

- Name
- Enrolment Number
- Subject
- Assignment Number
- Address

3. Get your assignment checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send your assignment to the National Institute of Open Schooling

1. Find the values of x and y if

$$(x-iy)+7-2i=9-i$$ (2)

2. Prove that the quadratic equation $x^2+py-4=0$ has real and distinct roots for all real values of p (2)

3. If $A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -2\\ 0 & 1\end{bmatrix}$,

find AB and BA. Is AB = BA? Is it true in general? (2)

4. If ${}^{18}C_r = {}^{18}C_{r+2}$ find the value of ${}^{r}C_5$ (2)

5. Show that $\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=(a+b+c)^3$ (3)

6. Using the principle of mathematical induction ,prove that

$$3^n \geq 2n+1$$

for any natural number n. (3)

7. If α, β are the roots of the equation $2x^2 - 6x + 5 = 0$, find the equation whose roots are

$$\alpha + \frac{1}{\beta} \text{ and } \beta + \frac{1}{\alpha}$$ (3)

8. Assuming that the expansions are possible, determine the coefficient of y^3 in

$$(1-4y)^2 (1-2y^2)^{\frac{1}{2}}.$$ (3)

9. Using Cramer's rule, solve the following system of equations:

$$\begin{aligned} x-2y &= 3+z \\ 2x+z &= y \\ 3-x &= 2z-3y \end{aligned}$$ (4)

10. Solve:

$$\frac{1}{3-2x} + \frac{5}{5+2x} = \frac{1}{2}$$ (4)

11. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can they be seated? (4)

12. Find the inverse of A= $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$ and verity that $A A^{-1} = I_3$. (6)

13. In a small scale industry, a menufacture produces two types of book cases. The first type of book case requires 3 hours on machine A and 2 hours on machine B for completion, whereas the second type of book case requires 3 hours on machine A and 3 hours on machine B. The machine A can run at the most for 18 hours while the machine B for at the most 14 hours per day. He earns a profit of Rs 30 on each book case of first type and Rs 40 on each book case of the second type.

How many book cases of each type should he make each day so as to maximise profit? What is the maximum profit? (6)

14. Assuming y to be so small that y^2 and higher powers of y can be neglected, find the value of the following:

$$\frac{(1-4y)^{-3}(1-2y^2)^{\frac{1}{2}}}{(4-y)^{\frac{3}{2}}}$$ (6)

MODULE-II

COORDINATE GEOMETRY

9. Cartesian System of Coordinates
10. Straight Lines
11. Circles
12. Conic Sections

9

CARTESIAN SYSTEM OF COORDINATES

You must have searched for your seat in a cinema hall, a stadium, or a train. For example, seat *H*-4 means the fourth seat in the H^{th} row. In other words, *H* and 4 are the coordinates of your seat. Thus, the geometrical concept of location is represented by numbers and alphabets (an algebraic concept).

Also a road map gives us the location of various houses (again numbered in a particular sequence), roads and parks in a colony, thus representing algebraic concepts by geometrical figures like straight lines, circles and polygons.

The study of that branch of Mathematics which deals with the interrelationship between geometrical and algebraic concepts is called Coordinate Geometry or Cartesian Geometry in honour of the famous French mathematician **Rene Descartes.**

In this lesson we shall study the basics of coordinate geometry and relationship between concept of straight line in geometry and its algebraic representation.

OBJECTIVES

After studying this lesson, you will be able to:

- define Cartesian System of Coordinates including the origin, coordinate axes, quadrants, etc;
- derive distance formula and section formula;
- derive the formula for area of a triangle with given vertices;
- verify the collinearity of three given points;
- state the meaning of the terms : inclination and slope of a line;

- find the formula for the slope of a line through two given points;
- state the condition for parallelism and perpendicularity of lines with given slopes;
- find the intercepts made by a line on coordinate axes;
- define locus as the path of a point moving in a plane under certain conditions; and
- find the equation of locus under given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Number system .
- Plotting of points in a coordinate plane.
- Drawing graphs of linear equations .
- Solving systems of linear equations .

9.1 RECTANGULAR COORDINATE AXES

Recall that in previous classes, you have learnt to fix the position of a point in a plane by drawing two mutually perpendicular lines. The fixed point O, where these lines intersect each other is called the **origin** O as shown in Fig. 9.1 These mutually perpencular lines are called the **coordinate axes**. The horizontal line XOX' is the **x-axis** or **axis** of x and the vertical line YOY' is the y- **axis** or **axis** of y.

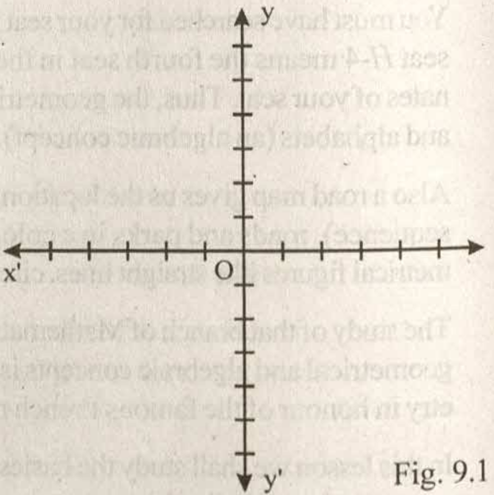


Fig. 9.1

9.1.1 CARTESIAN COORDINATES OF A POINT

To find the coordinates of a point we proceed as follows. Take X'OX and YOY' as coordinate axes. Let P be any point in this plane. From point P draw $PA \perp XOX'$ and $PB \perp YOY'$. Then the distance OA = x measured along x-axis and the distance OB = y measured along y-axis determine the position of the point P with reference to these axes. The distance OA measured along the axis of x is called the **abscissa** or x-coordinate and the distance OB (=PA) measured along y – axis is called the **ordinate** or y-coordinate of the point P. The abscissa and the ordinate taken together are called the **coordinates** of the point P. Thus, the coordinates of the point P are (x and y) which represent the position of the point P point in a plane. These two numbers are to form an **ordered pair** beacuse the order in which we write these numbers is important.

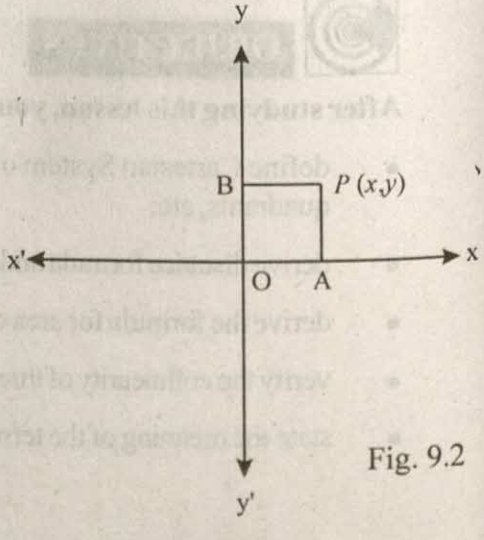


Fig. 9.2

In Fig. 9.3 you may note that the position of the ordered pair (3,2) is different from that of (2,3). Thus, we can say that (x,y) and (y,x) are two different ordered pairs representing two different points in a plane.

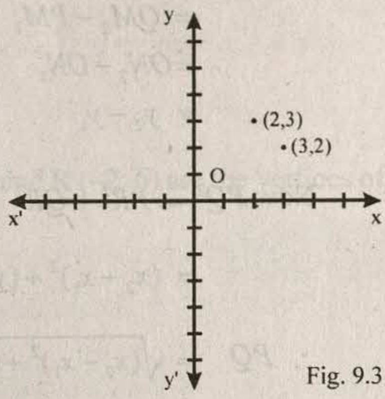


Fig. 9.3

9.1.2 QUARDRANTS

We know that coordinate axes XOX' and YOY' divide the region of the plane into four regions. These regions are called the quardrants as shown in Fig. 9.4. In accordance with the convention of signs, for a point P (x,y) in different quadrants, we have

I	quadrant :	$x > 0, y > 0$
II	quadrant :	$x < 0, y > 0$
III	quadrant :	$x < 0, y < 0$
IV	quadrant :	$x > 0, y < 0$

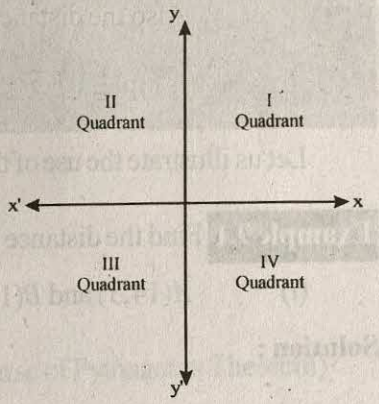


Fig. 9.4

9.2 DISTANCE BETWEEN TWO POINTS

Recall that you have derived the distance formula between two points P (x_1,y_1) and Q (x_2,y_2) in the following manner:

Let us draw a line $l \parallel XX'$ through P. Let R be the point of intersection of the perpendicular from Q to the line l. Then ΔPQR is a right-angled triangle.

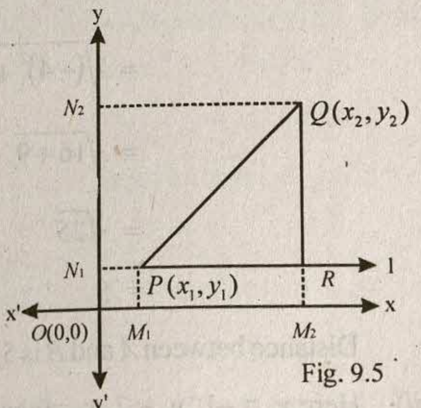


Fig. 9.5

$$\text{Also } PR = M_1M_2$$

$$= OM_2 - OM_1$$

$$= x_2 - x_1$$

$$\text{and } QR = QM_2 - RM_2$$

$$= QM_2 - PM_1$$
$$= ON_2 - ON_1$$
$$= y_2 - y_1$$

Now $PQ^2 = PR^2 + QR^2$ (Pythagoras theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \; PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note : This formula holds for points in all quadrants.

Also the distance of a point P(x,y) from the origin O$(0,0)$

is OP $= \sqrt{x^2 + y^2}$.

Let us illustrate the use of these formulae with some examples.

Example 9.1 Find the distance between the following pairs of points :

(*i*) $A(14,3)$ and $B(10,6)$ (*ii*) $M(-1,2)$ and $N(0,-6)$

Solution :

(*i*) Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $x_1 = 14, y_1 = 3, x_2 = 10, y_2 = 6$

$\therefore$ Distance between A and B $= \sqrt{(10-14)^2 + (6-3)^2}$

$$= \sqrt{(-4)^2 + (3)^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

Distance between A and B is 5 units.

(*ii*) Here $x_1 = -1, y_1 = 2, x_2 = 0$ and $y_2 = -6$

Distance between M and N $= \sqrt{(0-(-1))^2 + (-6-2)^2}$

$$= \sqrt{1+(-8)^2}$$

$= \sqrt{1+64} = \sqrt{65}$

Distance between M and N $= \sqrt{65}$ units

Example 9.2 Show that the points P(–1, –1), Q(2, 3) and R (–2, 6) are the vertices of a right-angled triangle.

Solution:

$PQ^2 = (2+1)^2 + (3+1)^2$

$= 3^2 + 4^2$

$= 9 + 16 = 25$

$QR^2 = (-4)^2 + (3)^2$

$= 16 + 9 = 25$

and $RP^2 = 1^2 + (-7)^2$

$= 1 + 49$

$= 50$

$\therefore \quad PQ^2 + QR^2 = 25+25=50= RP^2$

$\Rightarrow \quad \Delta PQR$ is a right-angled triangle (by converse of Pythagoras Theorem)

Example 9.3 Show that the points A(1, 2), B(4, 5) and C(–1, 0) lie on a straight line.

Solution: Here,

$AB = \sqrt{(4-1)^2 + (5-2)^2}$ units

$= \sqrt{18}$ units

$= 3\sqrt{2}$ units

$BC = \sqrt{(-1-4)^2 + (0-5)^2}$ units

$= \sqrt{50}$ units

$= 5\sqrt{2}$ units

and $AC = \sqrt{(-1-1)^2 + (0-2)^2}$ units

$= \sqrt{4+4}$ units

$= 2\sqrt{2}$ units

Now $AB + AC = \left(3\sqrt{2} + 2\sqrt{2}\right) units = 5\sqrt{2}\ units = BC$

i.e. BA + AC = BC

Hence, A, B, C lie on a straight line. In other words, A,B,C are collinear.

Example 9.4 Prove that the points (2a, 4a), (2a, 6a) and $\left(2a + \sqrt{3}a, 5a\right)$ are the vertices of an equilateral triangle whose side is 2a.

Solution: Let the points be A (2a, 4a), B (2a, 6a) and C $\left(2a + \sqrt{3}a, 5a\right)$

$AB = \sqrt{0 + (2a)^2}$ =2a units

$BC = \sqrt{(\sqrt{3}a)^2 + (-a)^2}$ units

$= \sqrt{3a^2 + a^2}$

= 2a units

and AC = $\sqrt{(\sqrt{3}a)^2 + (+a)^2}$ = 2a units

⇒ AB + BC > AC, BC + AC > AB and

AB + AC > BC and AB = BC = AC = 2a

⇒ A, B, C form the vertices of an equilateral triangle of side 2a.

CHECK YOUR PROGRESS 9.1

1. Find the distance between the following pairs of points.

   (a) (5, 4) and (2, –3) (b) (a, –a) and (b, b)

2. Prove that each of the following sets of points are the vertices of a right angled-trangle.

   (a) (4, 4), (3, 5), (–1,–1) (b) (2, 1), (0, 3), (–2, 1)

3. Show that the following sets of points form the vertices of a triangle:

   (a) (3, 3), (–3, 3) and (0, 0) (b) (0, a), (a, b) and (0, 0) (if ab = 0)

4. Show that the following sets of points are collinear :

   (a) (3, –6), (2, –4) and (–4, 8) (b) (0,3), (0, –4) and (0, 6)

5. (a) Show that the points (0, –1), (–2, 3), (6, 7) and (8, 3) are the vertices of a rectangle.

   (b) Show that the points (3, –2), (6, 1), (3, 4) and (0, 1) are the vertices of a square.

9.3 SECTION FORMULA

9.3.1 INTERNAL DIVISION

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points on a line l and $R(x, y)$ divide PQ internally in the ratio $m_1 : m_2$.

To find : The coordinates x and y of point R.

Construction : Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and L, M and N lie on XX'. Also draw $RT \perp QN$ and $PV \perp QN$.

Method : R divides PQ internally in the ratio $m_1 : m_2$.

$\Rightarrow$ R lies on PQ and $\dfrac{PR}{RQ} = \dfrac{m_1}{m_2}$

Also, in triangles, RPS and QRT,

$\angle RPS = \angle QRT$ (Corresponding angles as $PS \parallel RT$)

and $\angle RSP = \angle QTR = 90^0$

$\therefore \quad \Delta RPS \sim \Delta QRT$ (AAA similarity)

$$\Rightarrow \frac{PR}{RQ} = \frac{RS}{QT} = \frac{PS}{RT} \quad \ldots (i)$$

Also, $PS = LM = OM - OL = x - x_1$

$RT = MN = ON - OM = x_2 - x$

$RS = RM - SM = y - y_1$

$QT = QN - TN = y_2 - y$.

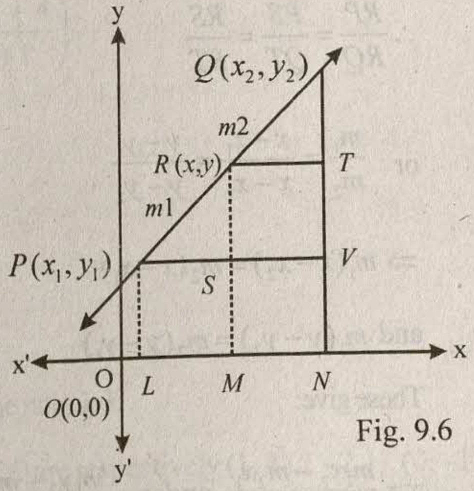


Fig. 9.6

From (i), we have

$$\therefore \frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow m_1(x_2 - x) = m_2(x - x_1)$$

and $m_1(y_2 - y) = m_2(y - y_1)$

$$\Rightarrow x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \quad and \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Thus, the coordinates of R are:

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$

Coordinates of the mid-point of a line segment

If R is the mid point of PQ, then,

$m_1 = m_2 = 1$ (as R divides PQ in the ratio 1:1

Coordinates of the mid point are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

9.3.2 EXTERNAL DIVISION

Let R divide PQ externally in the ratio $m_1 : m_2$

To find : The coordinates of R.

Construction : Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and $PS \perp RM$ and $QT \perp RM$.

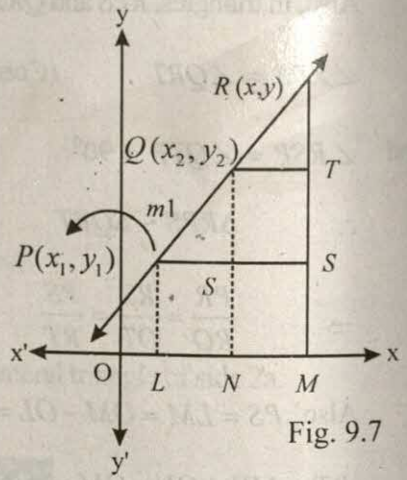


Fig. 9.7

Clearly, $\Delta RPS \sim \Delta RQT$.

$$\therefore \frac{RP}{RQ} = \frac{PS}{QT} = \frac{RS}{RT}$$

$$\text{or} \quad \frac{m_1}{m_2} = \frac{x-x_1}{x-x_2} = \frac{y-y_1}{y-y_2}$$

$$\Rightarrow m_1(x-x_2) = m_2(x-x_1)$$

and $m_1(y-y_2) = m_2(y-y_1)$

These give:

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \quad and \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

Hence, the coordinates of the point of external division are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

Let us now take some examples.

Example 9.5 Find the coordinates of the point which divides the line segment joining the points (4, –2) and (–3, 5) internally and externally in the ratio 2:3.

Solution:

(i) Let $P(x, y)$ be the point of internal division.

$$\therefore x = \frac{2(-3)+3(4)}{2+3} = \frac{6}{5} \text{ and } y = \frac{2(5)+3(-2)}{2+3} = \frac{4}{5}$$

P has coordinates $\left(\frac{6}{5}, \frac{4}{5}\right)$

If $Q(x', y')$ is the point of external division, then

$$x' = \frac{(2)(-3)-3(4)}{2-3} = 18 \text{ and } y' = \frac{(2)(5)-3(-2)}{2-3} = -16$$

Thus, the coordinates of the point of external division are (18, –16).

Example 9.6 In what ratio does the point (3, –2) divide the line segment joining the points (1,4) and (–3, 16) ?

Solution : Let the point $P(3, -2)$ divide the line segement in the ratio $k : 1$.

Then the coordinates of P are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But the given coordinates of P are (3, –2)

$$\therefore \frac{-3k+1}{k+1} = 3$$

$$\Rightarrow -3k+1 = 3k+3$$

$$\Rightarrow k = -\frac{1}{3}$$

$\Rightarrow$ P divides the line segement externally in the ratio 1:3.

Example 9.7 The vertices of a quadrilateral $ABCD$ are respectively (1, 4), (–2,1), (0,–1) and (3, 2). If E, F, G, H are respectively the midpoints of AB, BC, CD and DA, prove that the quadrilateral $EFGH$ is a parallelogram.

Solution : Since E, F, G, and H, are the midpoints of the sides AB, BC, CD and DA, therefore, the coordinates of E, F, G, and H respectively are :

$$\left(\frac{1-2}{2}, \frac{4+1}{2}\right), \left(\frac{-2+0}{2}, \frac{1-1}{2}\right), \left(\frac{0+3}{2}, \frac{-1+2}{2}\right) \text{ and } \left(\frac{1+3}{2}, \frac{4+2}{2}\right)$$

$$\Rightarrow E\left(\frac{-1}{2},\frac{5}{2}\right),\ F(-1,0), G\left(\frac{3}{2},\frac{1}{2}\right) \text{ and } H(2,3) \text{ are the required points.}$$

Also, the mid point of diagonal *EG* has coordinates

$$\left(\frac{\frac{-1}{2}+\frac{3}{2}}{2},\frac{\frac{5}{2}+\frac{1}{2}}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$$

Coordinates of midpoint of *FH* are

$$\left(\frac{-1+2}{2},\frac{0+3}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$$

Since, the midpoints of the diagonals are the same, therefore, the diagonals bisect each other.

Hence *EFGH* is a parallelogram.

CHECK YOUR PROGRESS 9.2

1. Find the midpoint of each of the line segements whose end points are given below:

   (a) (–2, 3) and (3, 5) (b) (6,0) and (–2,10)

2. Find the coordinates of the point dividing the line segment joining

   (–5,–2) and (3, 6) internally in the ratio 3:1.

3. (a) Three vertices of a parallelogram are (0,3), (0,6) and (2,9). Find the fourth vertex.

   (b) (4, 0), (–4, 0), (0,–4) and (0, 4) are the vertices of a square. Show that the quadrilateral formed by joining the midpoints of the sides is also a square.

4. The line segement joining (2, 3) and (5, –1) is trisected. Find the points of trisection.

5. Show that the figure formed by joining the midpoints of the sides of a rectangle is a rhombus.

9.4 AREA OF A TRIANGLE

Let us find the area of a triangle whose

vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

Draw *AL*, *BM* and *CN* perpendiculars to *XX'*.

area of Δ ABC

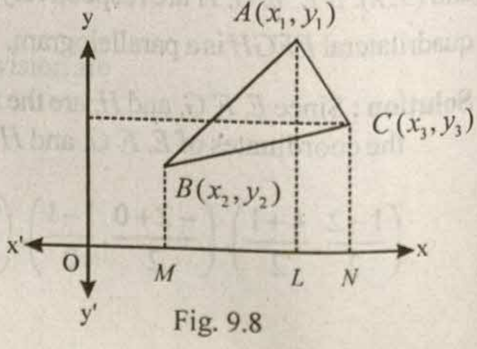


Fig. 9.8

= Area of trapzium. $BMLA$ + Area of trapzium. ALNC – Area of trapzium. BMNC

$$=\frac{1}{2}(BM+AL)ML+\frac{1}{2}(AL+CN)LN-\frac{1}{2}(BM+CN)MN$$

$$=\frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$$

$$=\frac{1}{2}\left[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_3y_3)\right]$$

$$=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

This can be stated in the determinant form as follows :

$$\text{Area of } \Delta\,\text{ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example 9.8 Find the area of the triangle whose vertices are $A(3, 4)$, $B(6, -2)$ and $C(-4, -5)$.

Solution: The area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 6 & -2 & 1 \\ -4 & -5 & 1 \end{vmatrix}$

$$=\frac{1}{2}\left[3(-2+5)-4(6+4)+1(-30-8)\right]$$

$$=\frac{1}{2}\left[9-40-38\right]=\frac{-69}{2}$$

As the area is to be positive

$\therefore$ Area of $\Delta ABC = \frac{69}{2}$ square units

Example 9.9 If the vertices of a triangle are $(1, k)$, $(4, -3)$ and $(-9, 7)$ and its area is 15 square units, find the value(s) of k.

Solution : Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & k & 1 \\ 4 & -3 & 1 \\ -9 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{2}[-3-7-k(4+9)+1(28-27)]$$

$$= \frac{1}{2}[-10-13k+1]$$

$$= \frac{1}{2}[-9-13k]$$

Since the area of the triangle is given to be15,

$$\therefore \quad \frac{-9-13k}{2} = 15$$

or, $\quad -9-13\text{k} = 30$

$\quad -13\text{k} = 39$

or, $\quad k = -3$

CHECK YOUR PROGRESS 9.3

1. Find the area of each of the following triangles whose vertices are given below :

   (1) (0, 5), (5, –5), and (0,0) (b) (2, 3), (–2,–3) and (–2, 3)

   (c) (a, 0), (0, – a) and (0, 0)

2. The area of a triangle ABC, whose vertices are A (2, –3), B(3, –2) and C$\left(\frac{5}{2}, k\right)$ is $\frac{3}{2}$ sq unit. Find the value of k

3. Find the area of a rectangle whose vertices are (5, 4), (5,– 4), (–5, 4) and (–5, – 4)

4. Find the area of a quadrilateral whose vertices are (5, –2), (4, –7), (1, 1) and (3, 4)

9.5 CONDITION FOR COLLINEARITY OF THREE POINTS

The three points $A(x_1, y_1),\ B(x_2, y_2)\ and\ C(x_3, y_3)$ are collinear if and only if the area of the triangle ABC becomes zero.

i.e. $$\frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3] = 0$$

i.e. $$x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$$

In short, we can write this result as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Let us illustrate this with the help of examples:

Example 9.10 Show that the points $A(a,\ b+c),\ B(b, c+a)\ and\ C(c, a+b)$ are collinear.

Solution : Area of triangle ABC $= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Hence the points are collinear.

Example 9.11 For what value of *k*, are the points (1, 5) , (k, 1) and (4, 11) collinear ?

Solution : Area of the triangle formed by the given points is

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ k & 1 & 1 \\ 4 & 11 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-10-5k+20+11k-4]$$

$$= \frac{1}{2}[6k+6] = 3k+3$$

Since the given points are collinear, therefore

$$3k + 3 = 0 \Rightarrow k = -1$$

Hence, for $k=-1$, the given points are collinear.

CHECK YOUR PROGRESS 9.4

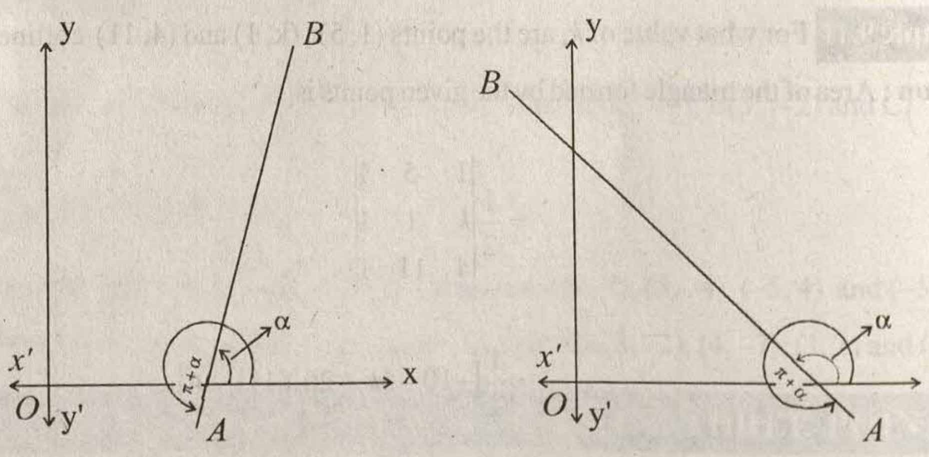
1. Show that the points (–1,–1), (5, 7) and (8, 11) are collinear.
2. Show that the points (3, 1), (5, 3) and (6, 4) are collinear.
3. Prove that the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear if $\frac{1}{a}+\frac{1}{b}=1$.
4. If the points $(a, b), (a_1, b_1)$ and $(a-a_1, b-b_1)$ are collinear, show that $a_l b = ab_l$
5. Find the value of k for which the points $(5, 7)$, $(k, 5)$ and $(0, 2)$ are collinear.
6. Find the values of k for which the point $(k, 2-2k)$, $(-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear.

9.6 INCLINATION AND SLOPE OF A LINE

Look at the Fig. 9.9. The line AB makes an angle α or $\pi+\alpha$ with the x-axis (measured in anticlockwise direction).

The *inclination* of the given line is represented by the measure of angle made by the line with the positive direction of x-axis (measured in anticlockwise direction)

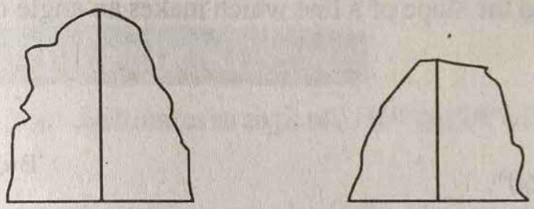
In a special case when the line is parallel to x-axis or it coincides with the *x-axis,* the inclination of the line is defined to be 0^0.



(a) (b)

Fig. 9.9

Again look at the pictures of two mountains given below. Here we notice that the mountain in Fig. 9.10 (a) is more steep compaired to mountain in Fig. 9.10 (b).



(a) Fig. 9.10 (b)

How can we quantify this steepness ? Here we say that the angle of inclination of mountain (a) is more than the angle of inclination of mountain (b) with the ground.

Try to see the difference between the ratios of the maximum height from the ground to the base in each case.

Naturally, you will find that the ratio in case (a) is more as compared to the ratio in case (b). That means we are concerned with height and base and their ratio is linked with tangent of an angle, so mathematically this ratio or the tangent of the inclination is termed as *slope.* We define the slope as tangent of an angle.

The *slope* of a line is the the tangent of the angle θ (say) which the line makes with the positive direction of x-axis. Generally, it is denoted by m $(= \tan\theta)$

Note : If a line makes an angle of 90^0 or 270^0 with the x–axis, the slope of the line can not be defined.

Example 9.12 In Fig. 9.9 find the slopes of lines *AB* and *BA.*

Solution : Slope of line $AB = \tan\alpha$

Slope of line $BA = \tan(\pi+\alpha) = \tan\alpha$.

Note : From this example, we can observe that "slope is independent of the direction of the line segement".

Example 9.13 Find the slope of a line which makes an angle of 30^0 with the negative direction of *x*-axis.

Solution :

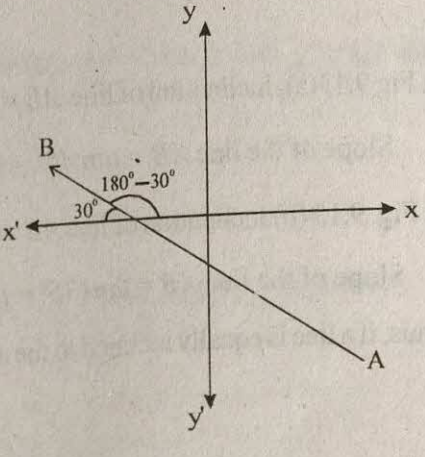


Fig. 9.11

Here $\theta = 180^0 - 30^0 = 150^0$

$$\therefore \quad m = \text{slope of the line}$$
$$= \tan(180^0 - 30^0)$$
$$= -\tan 30^0$$
$$= -\frac{1}{\sqrt{3}}$$

Example 9.14 Find the slope of a line which makes an angle of 60^0 with the positive direction of *y*-axis.

Solution :

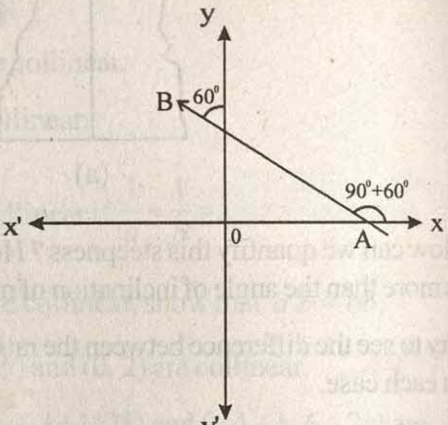


Fig. 9.12

Here $\theta = 90^0 + 60^0$

$\therefore \quad m$ = slope of the line

$= \tan(90^0 + 60^0)$

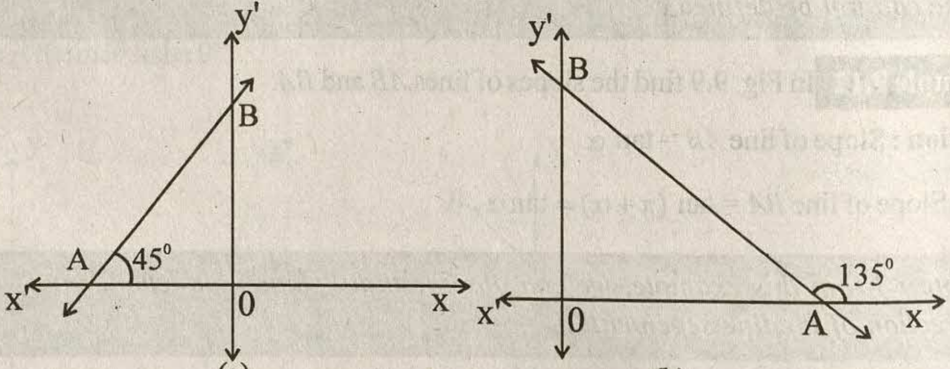
$= -\cot 60^0$

$= -\tan 30^0$

$= -\frac{1}{\sqrt{3}}$

Example : 9.15 If a line is equally inclined to the axes, show that its slope is ± 1.

Solution : Let a line *AB* be equally inclined to the axes and meeting axes at points *A* and *B* as shown in the Fig. 9.13



(a)

(b)

Fig. 9.13

In Fig 9.13(a), inclination of line $AB = \angle XAB = 45^0$

$\therefore$ Slope of the line $AB = \tan 45^0 = 1$

In Fig. 9.13 (b) inclination of line $AB = \angle XAB = 180^0 - 45^0 = 135^0$

$\therefore$ Slope of the line $AB = \tan 135^0 = \tan(180^0 - 45^0) = -\tan 45^0 = -1$

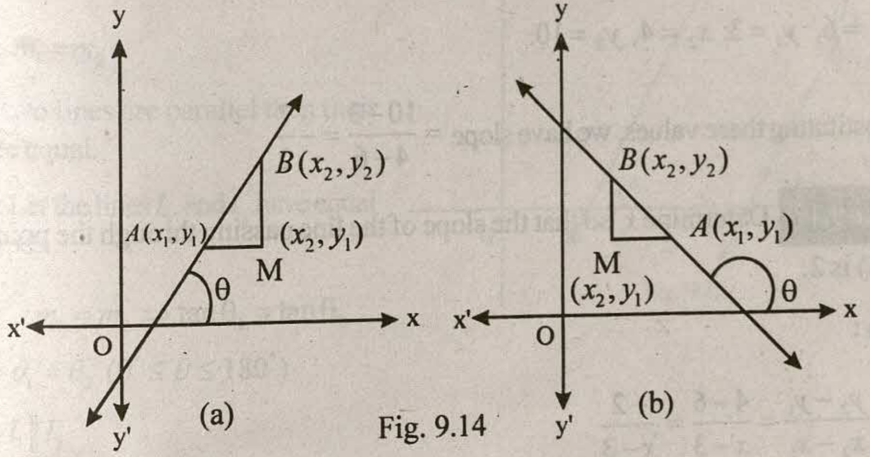
Thus, if a line is equally inclined to the axes, then the slope of the line will be ± 1.

CHECK YOUR PROGRESS 9.5

1. Find the Slope of a line which makes an angle of (i) 60°, (ii) 150° with the positive direction of x-axis.
2. Find the slope of a line which makes an angle of 30° with the positive direction of y-axis.
3. Find the slope of a line which makes an angle of 60° with the negative direction of x-axis.

9.7 SLOPE OF A LINE JOINING TWO DISTINCT POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points. Draw a line through A and B and let the inclination of this line be θ. Let the point of intersection of a horizontal line through A and a vertical line through B be M, then the coordinates of M are as shown in the Fig. 9.14



(a)

Fig. 9.14

(b)

(A) In Fig 9.14 (a), angle of inclination MAB is equal to θ (acute). Consequently.

$$\tan\theta = \tan(\angle MAB) = \frac{MB}{AM} = \frac{y_2 - y_1}{x_2 - x_1}$$

(B) In Fig. 9.14 (b), angle of inclination θ is obtuse, and since θ and $\angle MAB$ are supplementary, consequently,

$$\tan\theta = -\tan(\angle MAB) = -\frac{MB}{MA} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence in both the cases, the slope m of a line through $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note : If $x_1 = x_2$, then m is not defined. In that case the line is parallel to y-axis.

Is there a line whose slope is 1? Yes, when a line is inclined at 45^0 with the positive direction of x-axis.

Is there a line whose slope is $\sqrt{3}$? Yes, when a line is inclined at 60^0 with the positive direction of x-axis.

From the answers to these questions, you must have realised that given any real number m, there will be a line whose slope is m (because we can always find an angle α such that $\tan\alpha = m$).

Example 9.16 Find the slope of the line joining the points $A(6, 3)$ and $B(4, 10)$.

Solution : The slope of the line passing through the points (x_1, y_1) and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Here, $x_1 = 6,\ y_1 = 3;\ x_2 = 4,\ y_2 = 10.$

Now substituting these values, we have slope $= \dfrac{10-3}{4-6} = -\dfrac{7}{2}$

Example 9.17 Determine x, so that the slope of the line passing through the points (3, 6) and $(x, 4)$ is 2.

Solution :

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-6}{x-3} = \frac{-2}{x-3}$$

$$\therefore \frac{-2}{x-3} = 2 \qquad \text{.............. (Given)}$$

$$\therefore \quad 2x - 6 = -2 \quad or \quad x = 2$$

CHECK YOUR PROGRESS 9.6

1. What is the slope of the line joining the points $A(6, 8)$ and $B(4, 14)$?
2. Determine x so that 4 is the slope of the line through the points $A(6,12)$ and $B(x, 8)$.
3. Determine y, if the slope of the line joining the points $A(-8, 11)$ and $B(2, y)$ is $-\dfrac{4}{3}$.
4. $A(2, 3)$ $B(0, 4)$ and $C(-5, 0)$ are the vertices of a triangle ABC. Find the slope of the line passing through *the point B* and the mid point of AC

5. $A(-2,7)$, $B(1,0)$, $C(4,3)$ and $D(1,2)$ are the vertices of a quadrilateral $ABCD$. Show that

(i) slope of AB = slope of CD (ii) slope of BC = slope of AD

9.8 CONDITIONS FOR PARALLELISM AND PERPENDICULARITY OF LINES.

9.8.1 Slope of Parallel Lines

Let l_1, l_2, be two (non-vertical) lines with their slopes m_l and m_2 respectively.

Let θ_1 and θ_2 be the angles of inclination of these lines respectively.

Case I : Let the lines *l_l and l_2 be parallel*

Then $\theta_1 = \theta_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

Thus, if two lines are parallel then their slopes are equal.

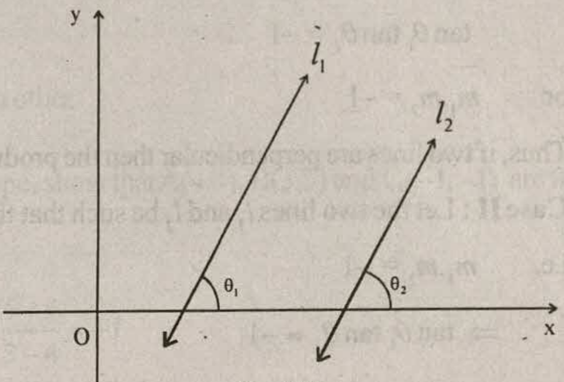


Fig. 9.15

Case II : Let the lines $l_{1'}$ and l_2 have equal slopes.

i.e. $m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

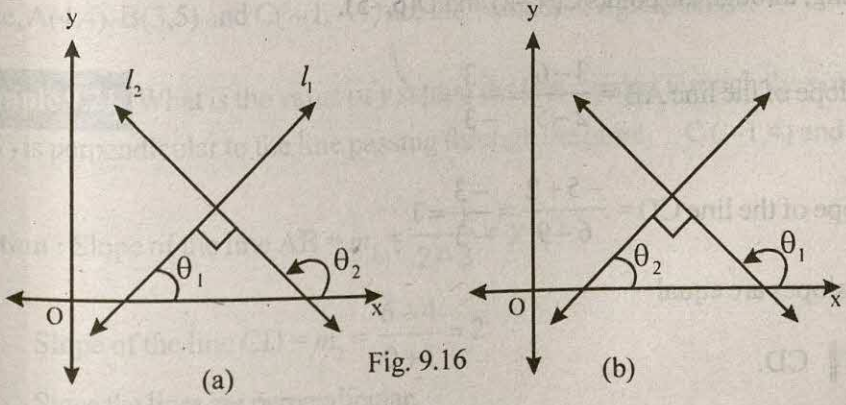
$\Rightarrow \theta_1 = \theta_2 \; (0^\circ \le \theta \le 180^\circ)$

$\Rightarrow l_1 \parallel l_2$

Hence, two (non-vertical) lines are parallel if and only if $m_1 = m_2$

9.8.2 SLOPES OF PERPENDICULAR LINES

Let l_1 and l_2 be two (non-vertical)lines with their slopes m_1 and m_2 respectively. Also let θ_1 and θ_2 be their inclinations respectively.



(a) (b)

Fig. 9.16

Case-I : Let $l_1 \perp l_2$

$$\Rightarrow \theta_2 = 90^0 + \theta_1 \quad \text{or} \quad \theta_1 = 90^0 + \theta_2$$

$$\Rightarrow \tan\theta_2 = \tan\left(90^0 + \theta_1\right) \quad \text{or} \quad \tan\theta_1 = \tan\left(90^0 + \theta_2\right)$$

$$\Rightarrow \tan\theta_2 = -\cot(\theta_1) \quad \text{or} \quad \tan\theta_1 = -\cot(\theta_2)$$

$$\Rightarrow \tan\theta_2 = -\frac{1}{\tan\theta_1} \quad \text{or} \quad \tan\theta_1 = -\frac{1}{\tan\theta_2}$$

$\Rightarrow$ In both the cases, we have

$$\tan\theta_1 \tan\theta_2 = -1$$

or $\quad m_1.m_2 = -1$

Thus, if two lines are perpendicular then the product of their slopes is equal to -1.

Case II : Let the two lines l_1 and l_2 be such that the product of their slopes is -1.

i.e. $\quad m_1.m_2 = -1$

$$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$$

$$\Rightarrow \tan\theta_1 = -\frac{1}{\tan\theta_2} = -\cot\theta_2 = \tan\left(90^0 + \theta_2\right)$$

$\Rightarrow$ Either $\theta_1 = 90^0 + \theta_2$ or $\theta_2 = 90^0 + \theta_1$

$\Rightarrow$ In both cases $l_1 \perp l_2$.

Hence, two lines are perpendicular if and only if $m_1.m_2 = -1$.

Example 9.18 Show that the line passing through the points A(5,6) and B(2,3) is parallel to the line passing, through the points C(9,–2) and D(6,–5).

Solution : Slope of the line AB $= \frac{3-6}{2-5} = \frac{-3}{-3} = 1$

and slope of the line CD $= \frac{-5+2}{6-9} = \frac{-3}{-3} = 1$

As the slopes are equal

$\therefore$ AB $\|$ CD.

Example 9.19 Show that the line passing through the points A(2,–5) and B(–2,5) is perpendicular to the line passing through the points L(6,3) and M(1,1).

Solution : Here

$$m_1 = \text{slope of the line AB} = \frac{5+5}{-2-2} = \frac{10}{-4} = \frac{-5}{2}$$

$$\text{and } m_2 = \text{slope of the line LM} = \frac{1-3}{1-6} = \frac{2}{5}$$

$$\text{Now } m_1 . m_2 = \frac{-5}{2} \times \frac{2}{5} = -1$$

Hence, the lines are perpendicular to each other.

Example 9.20 Using the concept of slope, show that A(4,4), B(3,5) and C$(-1,-1)$ are the vertices of a right triangle.

Solution : $\text{Slope of line AB} = m_1 = \dfrac{5-4}{3-4} = -1$

$$\text{Slope of line BC} = m_2 = \frac{-1-5}{-1-3} = \frac{3}{2}$$

$$\text{and slope of line AC} = m_3 = \frac{-1-4}{-1-4} = 1$$

Now $m_1 \times m_3 = -1$

$\Rightarrow \text{AB} \perp \text{AC}$

$\Rightarrow \triangle\text{ABC}$ is a right-angled triangle.

Hence, A(4,4), B(3,5) and C$(-1,-1)$ are the vertices of right triangle.

Example 9.21 What is the value of y so that the line passing through the points A(3,y) and B(2,7) is perpendicular to the line passing through the point C$(-1,4)$ and D (0,6)?

Solution : $\text{Slope of the line AB} = m_1 = \dfrac{7-y}{2-3} = y-7$

$$\text{Slope of the line CD} = m_2 = \frac{6-4}{0+1} = 2$$

Since the lines are perpendicular,

$$\therefore m_1 \times m_2 = -1$$

or $$(y-7)\times 2 = -1$$

or $$2y-14=-1$$

or $$2y=13$$

or $$y=\frac{13}{2}$$

CHECK YOUR PROGRESS 9.7

1. Show that the line joining the points (2,–3) and (–4,1) is
   (i) parallel to the line joining the points (7,–1) and (0,3).
   (ii) perpendicular to the line joining the points (4,5) and (0,–2).
2. Find the slope of a line parallel to the line joining the points (–4,1) and (2,3).
3. The line joining the points (–5,7) and (0,–2) is perpendicular to the line joining the points (1,3) and (4,x). Find x.
4. A(–2,7), B(1,0), C(4,3) and D(1,2) are the vertices of quadrilateral ABCD. Show that the sides of ABCD are parallel.
5. Using the concept of the slope of a line, show that the points A(6, – 1), B(5,0) and C(2,3) are collinear.[Hint: slopes of AB, BC and CA must be equal.]
6. Find k so that line passing through the points (k,9) and (2,7) is parallel to the line passing through the points (2,–2) and (6,4).
7. Using the concept of slope of a line, show that the points (–4,–1), (–2–4), (4,0) and (2,3) taken in the given order are the vertices of a rectangle.
8. The vertices of a triangle ABC are A(–3,3), B(–1,– 4) and C(5,–2). M and N are the midpoints of AB and AC. Show that MN is parallel to BC and MN = $\frac{1}{2}$ BC.

9.9 INTERCEPTS MADE BY A LINE ON AXES

If a line l (not passing through the Origin) meets x-axis at A and y-axis at B as shown in Fig. 9.17, then

(i) OA is called the x-intercept or the intercept made by the line on x-axis.

(ii) OB is called y-intercept or the intercept made by the line on y-axis.

(iii) OA and OB taken together in this order are called the intercepts made by the line l on the axes.

(iv) AB is called the portion of the line intercepted between the axes.

(v) The coordinates of the point A are $(a,0)$ and those of point B are $(0,b)$

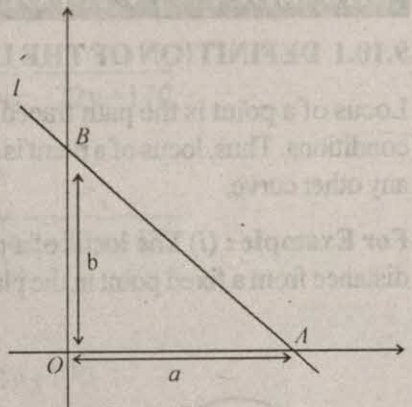


Fig. 9.17

To find the intercept of a line in a given plane on x-axis, we put $y=0$ in the given equation of a line and the value of x so obtained is called the x intercept.

To find the intercept of a line on y-axis we put $x=0$ and the value of y so obtained is called the y intercept.

Note: 1. A line which passes through origin makes no intercepts on axes.

2. A horizontal line has no x-intercept and vertical line has no y-intercept.

3. The intercepts on x- axis and y-axis are usually denoted by a and b respectively. But if only y-intercept is considered, then it is usually denoted by c.

Example 9.22 If a line is represented by $2x+3y=6$, find its x and y intercepts.

Solution : The given equation of the line is

$$2x+3y=6 \qquad \text{... } (i)$$

Putting $x=0$ in (i), we get

$$y=2$$

Thus, y-intercept is 2.

Again putting $y=0$ in (i), we get

$$2x=6 \Rightarrow x=3$$

Thus, x-intercept is 3.

CHECK YOUR PROGRESS 9.8

1. Find x and y intercepts, if the equations of lines are :

(*i*) $x+3y=6$ (*ii*) $7x+3y=2$ (*iii*) $\frac{x}{2a}+\frac{y}{2b}=1$ (*iv*) $ax+by=c$

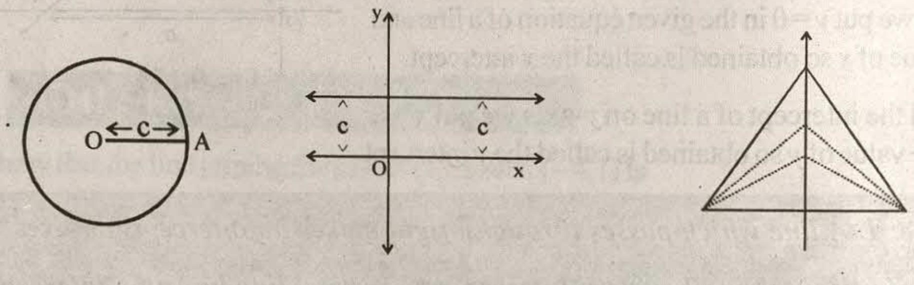
(*v*) $\frac{y}{2}-2x=8$ (*vi*) $\frac{y}{3}-\frac{2x}{3}=7$

9.10 LOCUS OF A POINT

9.10.1 DEFINITION OF THE LOCUS OF A POINT

Locus of a point is the path traced by the point when moving under a given condition or conditions. Thus, locus of a point is a path of definite shape. It may be a straight line, circle or any other curve.

For Example : (*i*) The locus of a point in a plane which moves such that it is at constant distance from a fixed point in the plane is a circle as shown in Fig. 9.18 (*a*).



(a) (b) (c)

Fig. 9.18

(*ii*) The locus of a point which moves such that it is always at a constant distance from *x*-axis is a pair of straight lines parallel to *x*-axis. [See Fig. 9.18 (b)]

(*iii*) The locus of a point in a plane which moves such that it is always at a constant distance from the two fixed points in the same plane is perpendicular bisector of the line segment joining the two points. [See Fig. 9.18(c)].

From the definition and the examples of a locus, we observe that

(a) Every point which satisfies the given condition or conditions is a point on the locus.

(b) Every point of the locus must satisfy the given condition or conditions.

9.10.2 EQUATION OF LOCUS

The equation of locus of a moving point (x,y) is an algebraic relation between x and y satisfying the given conditions of motion of a point.

The coordinates (x,y) of the moving point which generates the locus are called current coordinates. The point covers all the positions on the locus and is called the general point.

Let us take an example :

Let P(4,3) and Q(7,11) be two points. Let us try to locate a point R which is equidistant from both the points P and Q.

Let the Coordinates of R be (x,y).

Then $$PR = \sqrt{(x-4)^2+(y-3)^2} = \sqrt{x^2+y^2-8x-6y+25}$$

$$QR = \sqrt{(x-7)^2+(y-11)^2} = \sqrt{x^2+y^2-14x-22y+170}$$

$$\because\ PR = QR$$

$$\therefore\ \sqrt{x^2+y^2-8x-6y+25} = \sqrt{x^2+y^2-14x-22y+170}$$

Squaring, we get

$$x^2+y^2-8x-6y+25 = x^2+y^2-14x-22y+170$$

$$\Rightarrow 6x+16y-145=0$$

This is called the equation of the locus of a point R which is equidistant from the points P and Q.

From the above, we observe the following working rule:

9.10.3 WORKING RULE TO FIND THE EQUATION OF THE LOCUS OF A POINT

(*i*) Take any point (x,y) on the locus.

(*ii*) Write the given geometrical form in the terms of x and y and known constant or constants and simplify it, if necessary.

(*iii*) Express the given condition in mathematical form in the terms of x and y and known constant or constants and simplify it, if necessary.

(*iv*) The equation so obtained is the equation of the required locus.

Example 9.23 Find the equation of locus of points which are thrice as far from $(-a,0)$ as from $(a,0)$.

Solution : Let $P(x,y)$ be any point on locus. Also let $A(-a,0)$ and $B(a,0)$ be the two given points.

Then by the given condition.

$$PA = 3\ PB$$

$$\text{i.e. } PA^2 = 9\ PB^2$$

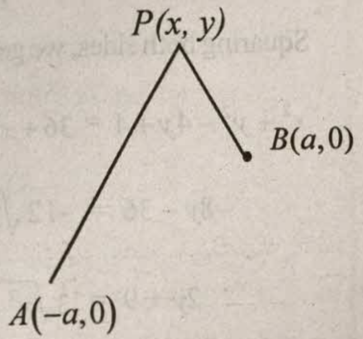


Fig. 9.19

$$\Rightarrow (x+a)^2+y^2=9\left[(x-a)^2+y^2\right]$$

$$\Rightarrow x^2+2ax+a^2+y^2=9\left[x^2-2ax+a^2+y^2\right]$$

$$\Rightarrow 8x^2-20ax+8a^2+8y^2=0$$

Thus, $2x^2+2y^2-5ax+2a^2=0$ is the required equation of the locus.

Example 9.24 Find the equation of the locus of a point such that the sum of its distances from (0,2) and (0,–2) is 6.

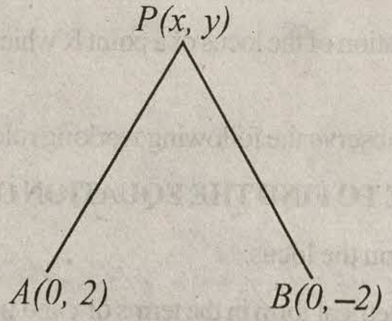


Fig. 9.20

Solution : Let P(x,y) be any point on the locus. Also let A(0,2) and B(0,–2) be the given points.

From the given condition, we have

$$PA+PB=6$$

$$\therefore \sqrt{x^2+(y-2)^2}+\sqrt{x^2+(y+2)^2}=6$$

or $\sqrt{x^2+y^2-4y+4}=6-\sqrt{x^2+y^2+4y+4}$

Squaring both sides, we get

$$x^2+y^2-4y+4=36+x^2+y^2+4y+4-12\sqrt{x^2+y^2+4y+4}$$

or $-8y-36=-12\sqrt{x^2+y^2+4y+4}$

or $2y+9=3\sqrt{x^2+y^2+4y+4}$

Squaring both sides again, we get

$$(2y+9)^2=9\left(x^2+y^2+4y+4\right)$$

or $4y^2+36y+81 = 9x^2+9y^2+36y+36$

or $9x^2+5y^2=45$

which is the required equation of the locus.

Example 9.25 A(3,1) and B(–2,4) are the two vertices of a triangle ABC. Find the equation of the locus of the centroid of the triangle, if the third vertex C is a point of the locus whose equation is $3x-4y=8$.

Solution : Let C(a,b) be the third vertex of the triangle ABC. Since C(a,b) lies on the locus whose equation is $3x-4y=8$.

$$\therefore 3a-4b=8 \qquad \ldots (i)$$

Let G(h,k) be the centroid of the triangle ABC.

$$\therefore h=\frac{3-2+a}{3} \text{ and } k=\frac{1+4+b}{3}$$

$$\Rightarrow a=3h-1 \text{ and } b=3k-5$$

Substituting these values of a and b in *(i)*, we get

$$3(3h-1)-4(3k-5)=8$$

$$\Rightarrow 3h-4k+3=0$$

Hence, the locus of the point G (h,k) is $3x-4y+3=0$

Example 9.26 (2,–2) is a point of the locus whose equation is $y^2=ax$. If (8,b) is also a point of locus, find b.

Solution : Since (2,–2) is a point of the locus whose equation is $y^2=ax$

$$\therefore (-2)^2=2a \Rightarrow a=2$$

$\therefore$ The equation of the locus is $y^2=2x$.

As (8,b) is also a point of this locus

$$\therefore b^2=2\times 8$$

$$\Rightarrow b^2=16 \Rightarrow b=\pm 4$$

Hence, the value of $b=\pm 4$

CHECK YOUR PROGRESS 9.9

1. Find the locus of a point which is equidistant from the points (3,4) and (–4,6).
2. Find the locus of a point equidistant from the points (4,2) and the *x*-axis.
3. Find the equation of locus of a point which moves so that the distance from the point (4,1) is twice its distance from the point (1,5).
4. A (2,3) and B (0,2) are the coordinates of the two vertices of a triangle. Find the locus of a point P such that the area of the triangle PAB = 3 sq. units.
5. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from the point (2,3) and (– 3,4) is 16.
6. Find the locus of a point which moves such that the sum of its distances from the points (3,0) and (–3,0) is less than 9.
7. If (*h*,0) is a point of the locus whose equation is $x^2 + y^2 - 6x + 8y - 36 = 0$, find *h*.
8. If $\left(\frac{3}{2},-2\right)$ is a point of the locus whose equation is $y^2 = ax$, find *b* if (*b*,6) is also a point of the locus.

LET US SUM UP

- Distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Coordinates of the point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$ are

  $$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

- Coordinates of the point dividing the line segment joining the the points (x_1, y_1) and (x_2, y_2) externally are in the ratio $m_1 : m_2$ are.

  $$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

- Coordinates of the mid point of the line segment joining the points (x_1,y_1) and (x_2,y_2) are

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The area of a triangle with vertices (x_1, y_1) and (x_2, y_2) and (x_3, y_3) is given by

  $$\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

- Three points A,.B, and C are collinear if the area of the triangle formed by them is zero.
- If θ is the angle which a line makes with the positive direction of *x*-axis, then the slope of the line is $m = \tan\theta$.
- Slope (m) of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

  $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- A line with the slope m_1 is parallel to the line with slope m_2 if $m_1 = m_2$.
- A line with the slope m_1 is perpendicular to the line with slope m_2 if $m_1 \times m_2 = -1$.
- If a line *l* (not passing through the origin) meets *x*- axis at A and y- axis at B then OA is called the *x*- intercept and OB is called the y- intercept.
- Locus of a point is the path traced by it when moving under given condition or conditions.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the distance between the pairs of points:

   (a) (2, 0) and (1, cot θ) (b) (–sin A, cosA) and (sin B, cos B)

2. Which of the following sets of points form a triangle?

   (a) (3, 2), (–3, 2) and (0, 3) (b) (3, 2), (3, –2) and (3, 0)

3. Find the midpoint of the line segment joining the points (3.–5) and (–6, 8).

4. Find the area of the triangle whose vertices are:

   (a) (1, 2), (–2, 3), (–3, –4) (b)(c, a), (c + a, a), (c – a, –a)

5. Show that the following sets of points are collinear (by showing that area formed is 0).

   (a) (–2, 5), (2, –3) and (0, 1) (b) (a, b + c), (b, c + a) and (c, a + b)

6. If (–3, 12), (7, 6) and (x, a) are collinear, find x.

7. Find the area of the quadrilateral whose vertices are (4,3) (–5,6) (0,7) and (3,–6).

8. Find the slope of the line through the points

   (a) (1,2), (4,2) (b) (4, –6), (–2, –5)

9. What is the value of y so that the line pasing through the points (3, y) and (2,7) is parallel to the line passing through the points (–1, 4) and (0, 6) ?

10. Without using Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right-angled triangle.

11. Using the concept of slope, determine which of the following sets of points are collnear:

    (i) (–2, 3) , (8, –5) and (5, 4), (ii) (5, 1) , (1, – 1) and (11, 4),

12. If A (2, –3) and B (3, 5) are two vertices of a rectangle ABCD, find the slope of

    (i) BC (ii) CD (iii) DA.

13. A quadrilateral has vertices at the points (7, 3), (3, 0), (0, –4) and (4, –1). Using slopes, show that the mid-points of the sides of the quadrilatral form a parallelogram.

14. Find the x-intercepts of the following lines:

    (*i*) $2x-3y=8$ (*ii*) $3x-7y+9=0$ (*iii*) $x-\frac{y}{2}=3$

15. Find the equation of the locus of a point equidistant from the points (2,4) and y-axis.

16. Find the equation of the locus of a point which is equidistant from the points (a+b, a–b) and (a–b, a+b).

17. If A(a,0), B(–a,0) are two fixed points, find the locus of a point P which moves so that $3|PA|=2|PB|$.

18. Find the equation of the locus of a point P if the sum of squares of its distances from (1,2) and (3,4) is 25 units.

ANSWERS

CHECK YOUR PROGRESS 9.1

(a) $\sqrt{58}$ (b) $\sqrt{2(a^2+b^2)}$

CHECK YOUR PROGRESS 9.2

1. (a) $\left(\frac{1}{2}, 4\right)$ (b) (2,5) 2. (1,4)

3. (a) (2,6) 4. $\left(3, \frac{5}{3}\right), \left(4, \frac{1}{3}\right)$

CHECK YOUR PROGRESS 9.3

1. (a) $\frac{25}{2}$ sq. units (b) 12 sq. units (c) $\frac{a^2}{2}$ sq. units

2. $k = \frac{5}{3}$ 3. 80 sq. units 4. $\frac{41}{2}$ sq. units

CHECK YOUR PROGRESS 9.4

5. $k = 3$ 6. $k = \frac{1}{2}, -1$

CHECK YOUR PROGRESS 9.5

1. (*i*) $\sqrt{3}$ (*ii*) $-\frac{1}{\sqrt{3}}$ 2. $-\sqrt{3}$ 3. $-\sqrt{3}$

CHECK YOUR PROGRESS 9.6

1. -3 2. 5 3. $-\frac{7}{3}$ 4. $\frac{5}{3}$

CHECK YOUR PROGRESS 9.7

2. $\frac{1}{3}$ 3. $\frac{14}{3}$. 6. $k = \frac{10}{3}$

CECK YOUR PROGRESS 9.8

1. (*i*) x-intercept $= 6$ (*ii*) x-intercept $= \frac{2}{7}$ (*iii*) x-intercept $= 2a$

y-intercept $= 2$ y-intercept $= \frac{2}{3}$ y-intercept $= 2b$

(*iv*) x-intercept $= \frac{c}{a}$ (*v*) x-intercept $= -4$ (*vi*) x-intercept $= \frac{-21}{2}$

y-intercept $= \frac{c}{b}$ y-intercept $= 16$ y-intercept $= 21$

CHECK YOUR PROGRESS 9.9

1. $14x - 4y + 27 = 0$ 2. $x^2 - 8x - 4y + 20 = 0$

3. $3x^2 + 3y^2 - 38y + 87 = 0$ 4. $x - 2y = 2$

5. $x^2 + y^2 + x - 7y + 11 = 0$ 6. $20x^2 + 36y^2 < 405$

7. $3 \pm 3\sqrt{5}$ 8. $\frac{27}{2}$

TERMINAL EXERCISE

1. (a) cosec θ (b) $2 \sin \frac{A+B}{2}$

2. None of the given sets forms a triangle.

3. $\left(-\frac{3}{2}, \frac{3}{2}\right)$ 4. (a) 11 sq. unit (b) a^2 sq. unit.

6. $\frac{51-5a}{3}$ 7. 29 sq. unit. 8. (*a*) 0 (b) $-\frac{1}{6}$

9. $y = 3$ 11. Only (*ii*) 12. (*i*) $-\frac{1}{8}$ (*ii*) 8 (*iii*) $-\frac{1}{8}$

14. (*i*) 4 (*ii*) -3 (*iii*) 3 15. $y^2 - 8y - 4x + 20 = 0$

16. $x - y = 0$ 17. $5x^2 + 5y^2 - 26ax + 5a^2 = 0$

18. $2x^2 + 2y^2 - 8x - 12y + 5 = 0$

10

STRAIGHT LINES

We have read about lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions. We have also seen that graphs of linear equations, which came out to be straight lines

Interestingly, the reverse problem of the above is finding the equations of straight lines, under different conditions, in a plane. The analytical geometry, more commonly called coordinate geomatry, comes to our help in this regard. In this lesson. We shall find equations of a straight line in different forms and try to solve problems based on those.

OBJECTIVES

After studying this lesson, you will be able to :

- derive equations of a line parallel to either of the coordinate axes;
- derive equations in different forms (slope-intercept, point-slope, two point, intercept, parametric and perpendicular) of a line;
- find the equation of a line in the above forms under given conditions;
- state that the general equation of first degree represents a line;
- express the general equation of a line into

  (i) slope-intercept form (ii) intercept form and (iii) perpendicular form;
- derive the formula for the angle between two lines with given slopes;
- find the angle between two lines with given slopes;
- derive the conditions for parallelism and perpendicularity of two lines;
- determine whether two given lines are parallel or perpendicular;
- derive an expression for finding the distance of a given point from a given line;
- calculate the distance of a given point from a given line;

- derive the equation of a line passing through a given point and parallel/perpendicular to a given line;
- write the equation of a line passing through a given point and:

  (i) parallel or perpendicular to a given line (ii) with given x-intercept or y-intercept (iii) passing through the point of intersection of two lines; and
- prove various geometrical results using coordinate geometry.

EXPECTED BACKGROUND KNOWLEDGE

- Congruence and similarity of traingles

10.1 STRAINGHT LINE PARALLEL TO AN AXIS

If you stand in a room with your arms stretched, we can have a line drawn on the floor parallel to one side. Another line perpendicular to this line can be drawn intersecting the first line between your legs.

In this situation the part of the line in front of you and going behind you is the y-axis and the one being parallel to your arms is the x-axis.

The direction part of the y-axis in front of you is positive and behind you is negative.

The direction of the part x-axis to your right is positive and to that to your left is negative.

Now, let the side facing you be at b units away from you, then the equation of this edge will be $y = b$ (parallel to x-axis)

where b is equal in absolute value to the distance from the x-axis to the opposite side.

If $b > 0$, then the line lies in front of you, i.e., above the x-axis.

If $b < 0$, then the line lies behind you, i.e., below the x-axis.

If $b = 0$, then the line passes through you and is the x-axis itself.

Again, let the side of the right of you is at c units apart from you, then the equation of this line will be $x = c$ (parallel to y - axis)

where c is equal in absolute value, to the distance from the y-axis on your right.

If $c > 0$, then the line lies on the right of you, i.e., to the right of y-axis.

If $c < 0$, then the line lies on the left of you, i.e., to the left of y-axis

If $c = 0$, then the line passes through you and is the y-axis.

Example 10.1 Find the equation of the lines passing through (2, 3) and is

(i) parallel to x-axis. (ii) parallel to y-axis.

Solution :

(i) The equation of any line parallel to x-axis is $y = b$

Since it passes through (2, 3), hence $b = 3$

$\therefore$ The required equation of the line is $y = 3$

(ii) The equation of any line parallel to y-axis is $x = c$

Since it passes through (2, 3), hence $c = 2$

$\therefore$ The required equation of the line is $x = 2$.

Example 10.2 Find the equation of the line passing through (–2, –3) and

(i) parallel to x-axis (ii) parallel to y-axis

Solution :

(i) The equation of any line parallel to x-axis is $y = b$

Since it passes through (–2, –3), hence $-3 = b$

$\therefore$ The required equation of the line is $y = -3$

(i) The equation of any line parallel to y-axis is $x = c$

Since it passes through (–2, –3), hence $-2 = c$

$\therefore$ The required equation of the line is $x = -2$

CHECK YOUR PROGRESS 10.1

1. If we fold and press the paper then what will the crease look like
2. Find the slope of a line which makes an angle of
   (a) 45° with the positive direction of x-axis.
   (b) 45° with the positive direction of y-axis.
   (c) 45° with the negative direction of x-axis.
3. Find the slope of a line joining the points (2, –3) and (3, 4).
4. Determine x so that the slope of the line through the points (2,5) and (7, x) is 3.
5. Find the equation of the line passing through (–3, –4) and
   (a) parallel to x-axis. (b) parallel to y-axis.
6. Find the equation of a line passing through (5, –3) and perpendicular to x-axis.
7. Find the equation of the line passing through (–3, –7) and perpendicular to y-axis.

10.2 DERIVATION OF THE EQUATION OF STRAIGHT LINE IN VARIOUS STANDARD FORMS

So far we have studied about the inclination, slope of a line and the lines parallel to the axes. Now the questions is, can we find a relationship between *x* and *y*, where (*x*, *y*) is any arbitrary point on the line?

The relationship between *x* and *y* which is satisfied by the co-ordinates of arbitrary point on the line is called the equation of a straight line. The equation of the line can be found in various forms under the given conditions, such as

(a) When we are given the slope of the line and its intercept on *y*-axis.

(b) When we are given the slope of the line and it passes through a given point.

(c) When the line passes through two given points.

(d) When we are given the intercepts on the axes by the line.

(e) When we are given the length of perpendicular from origin on the line and the angle which the perpendicualr makes with the positive direction of *x*-axis.

(f) When the line passes through a given point making an angle α with the positive direction of *x*-axis. (Parametric form).

We will discuss all the above cases one by one and try to find the equation of line in its standard forms.

(A) SLOPE-INTECEPT FORM

Let AB be a straight line making an angle θ with *x*-axis and cutting off an intercept $OD = c$ from OY.

As the line makes intercept OD = c on *y*-axis, it is called *y*-intercept.

Let AB intersect OX' at T.

Take any point $P(x, y)$ on AB. Draw $PM \perp OX$.

The $OM = x$, $MP = y$.

Draw $DN \perp MP$.

From the right-angled triangle DNP, we have

$$\tan\theta = \frac{NP}{DN} = \frac{MP - MN}{OM}$$

$$= \frac{y - OD}{OM}$$

$$= \frac{y - c}{x}$$

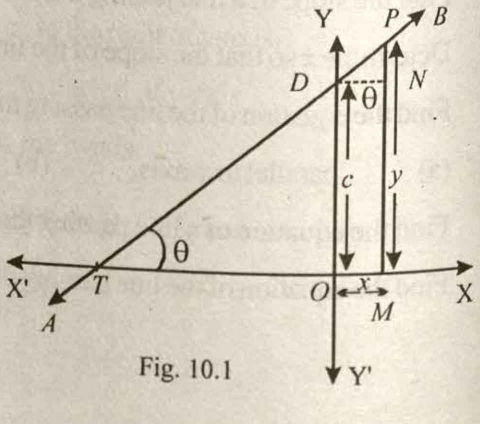


Fig. 10.1

$\therefore \quad y = x \tan\theta + c$

$\tan\theta = m$ (slope)

$\therefore \quad y = mx + c$

Since, this equation is true for every point on *AB,* and clearly for no other point in the plane, hence it represents the equation of the line *AB*.

Note : (1) When $c = 0$ and $m \neq 0 \Rightarrow$ the line passes through the origin and its equation is $y = mx$

(2) When $c = 0$ and $m = 0 \Rightarrow$ the line coincides with x – axis and its equation is of the form $y = 0$

(3) When $c \neq 0$ and $m = 0 \Rightarrow$ the line is parallel to x-axis and its equation is of the form $y = c$

Example 10.3 Find the equation of a line with slope 4 and y-intercept 0.

Solution : Putting $m = 4$ and $c = 0$ in the slope intercept form of the equation, we get $y = 4x$

This is the desired equation of the line.

Example 10.4 Determine the slope and the y-intercept of the line whose equation is $8x + 3y = 5$.

Solution : The given equation of the line is $8x + 3y = 5$

or, $$y = -\frac{8}{3}x + \frac{5}{3}$$

Comparing this equation with the equation $y = mx + c$ (Slope intercept form) we get

$$m = -\frac{8}{3} \text{ and } c = \frac{5}{3}$$

Therefore, slope of the line is $-\frac{8}{3}$ and its

y-intercept is $\frac{5}{3}$.

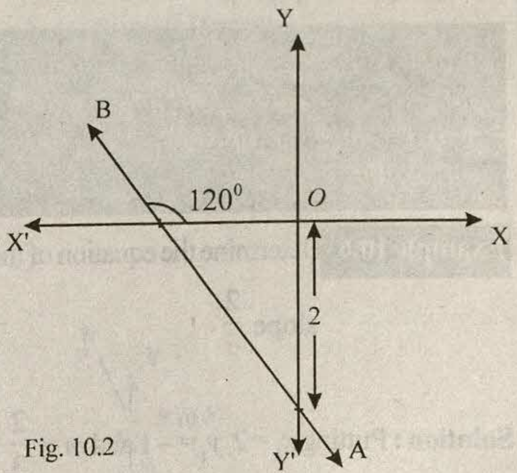


Fig. 10.2

Example 10.5 Find the equation of the line cutting off an intercept of length 2 from the negative direction of the axis of y and making an angle of 120^0 with the positive direction x-axis

Solution : From the slope intercept form of the line

$\therefore \quad y = x \tan 120^\circ + (-2)$

$$= -\sqrt{3}\,x - 2$$

or, $y + \sqrt{3}\,x + 2 = 0$

Here $m = \tan 120°$, and $c = -2$, because the intercept is cut on the negative side of y-axis.

(b) POINT-SLOPE FORM

Here we will find the equation of a line passing through a given point $A(x_1, y_1)$ and having the slope m.

Let $P(x, y)$ be any point other than A on given the line. Slope $(\tan\theta)$ of the line joining $A(x_1, y_1)$ and $P(x, y)$ is given by

$$\text{m} = \tan\theta = \frac{y - y_1}{x - x_1}$$

The slope of the line AP is given to be m.

$$\therefore \quad m = \frac{y - y_1}{x - x_1}$$

$\therefore$ The equation of the required line is

$$y - y_1 = m(x - x_1)$$

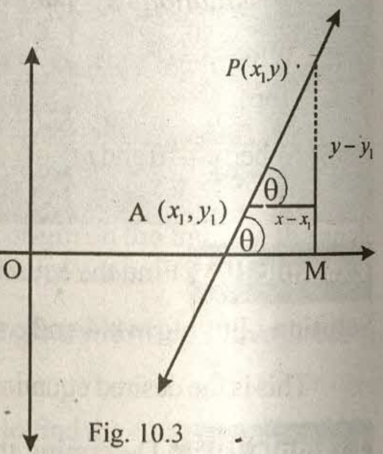


Fig. 10.3

Note: Since, the slope m is undefined for lines parallel to y-axis, the point-slope form of the equation will not give the equation of a line though A (x₁, y₁) parallel to y-axis. However this presents no difficulty, since for any such line the abscissa of any point on the line is x₁. Therefore the equation of such a line is x = x₁.

Example 10.6 Determine the equation of the line passing through the point (2,– 1) and having slope $\frac{2}{3}$.

Solution : Putting $x_1 = 2, y_1 = -1$ and $m = \frac{2}{3}$ in the equation of the point-slope form of the line we get

$$y - (-1) = \frac{2}{3}(x - 2)$$

$$\Rightarrow \quad y + 1 = \frac{2}{3}(x - 2)$$

$$\Rightarrow \quad y = \frac{2}{3}x - \frac{7}{3}$$

which is the required equation of the line.

(c) TWO POINT FORM

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given distinct points.

Slope of the line passing through these points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_2 \neq x_1)$$

From the equation of line in point slope form, we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the required equation of the line in two-point form.

Example 10.7 Find the equation of the line passing through (3, – 7) and (– 2,– 5).

Solution : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \cdots \text{(i)}$$

Since $x_1 = 3, y_1 = -7$ and $x_2 = -2$, and $y_2 = -5$, equation (i) becomes,

$$y + 7 = \frac{-5 + 7}{-2 - 3}(x - 3)$$

$$\text{or,} \quad y + 7 = \frac{2}{-5}(x - 3)$$

$$\text{or,} \quad 2x + 5y + 29 = 0$$

(d) INTERCEPT FORM

We want to find the equation of a line which cuts off given intercepts on both the co-ordinate axes.

Let PQ be a line meeting x-axis in A and y-axis in B. Let $OA = a$, $OB = b$.

Then the co-ordinates of A and B are $(a,0)$ and $(0, b,)$ respectively.

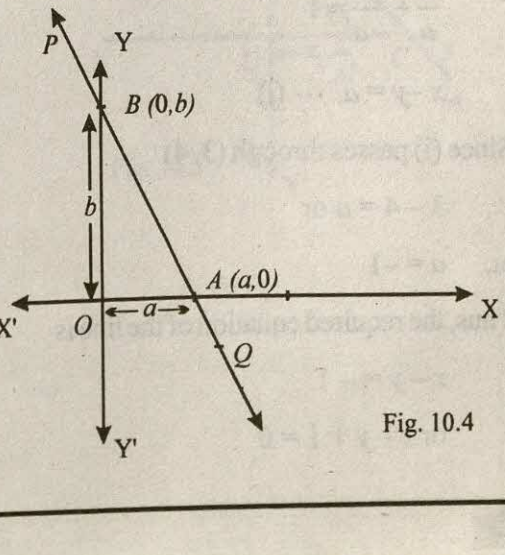


Fig. 10.4

The equation of the line joining A and B is

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

or, $$y = -\frac{b}{a}(x-a)$$

or, $$\frac{y}{b} = -\frac{x}{a} + 1$$

or, $$\frac{x}{a} + \frac{y}{b} = 1$$

This is the required equation of the line having intercepts a and b on the axes.

Example 10.8 Find the equation of a line which cuts off intercepts 5 and –3 on x and y axes respectively.

Solution : The intercepts are 5 and –3 on x and y axes respectively. i.e., $a = 5, b = -3$

The required equation of the line is

$$\frac{x}{5} + \frac{y}{-3} = 1$$

$$3x - 5y - 15 = 0$$

Example 10.9 Find the equation of a line which passes through the point (3, 4) and makes intercepts on the axes equl in magnitude but opposite in sign.

Solution : Let the x-intercept and y-intercept be a and –a respectively

$\therefore$ The equation of the line is

$$\frac{x}{a} + \frac{y}{-a} = 1$$

$$x - y = a \quad \cdots \text{(i)}$$

Since (i) passes through (3, 4)

$\therefore$ $3 - 4 = a$ or

or, $a = -1$

Thus, the required equation of the line is

$$x - y = -1$$

or $x - y + 1 = 0$

Example 10.10 Determine the equation of the line through the point (– 1,1) and parallel to x - axis.

Solution : Since the line is parallel to x-axis its slope ia zero. Therefore from the point slope form of the equation, we get

$$y-1=0\,[\,x-(-1)]$$

$$y-1=0$$

which is the required equation of the given line

Example 10.11 Find the intercepts made by the line

$3x-2y+12=0$ on the coordinate axes

Solution : Equation of the given line is

$$3x-2y=-12.$$

Dividing by – 12, we get

$$\frac{x}{-4}+\frac{y}{6}=1$$

Comparing it with the standard equation of the line in intercept form, we find $a=-4$ and $b=6$. Hence the intercepts on the x-axis and y-axis repectively are –4. and 6.

Example 10.12 The segment of a line, intercepted between the coordinate axes is bisected at the point (x_1, y_1). Find the equation of the line

Solution : Let $P(x_1, y_1)$ be the middle point or the segment CD of the line AB intercepted between the axes. Draw $PM \perp OX$

$\therefore \quad OM = x_1$ and $MP = y_1$

$\therefore \quad OC = 2x_1$ and $OD = 2y_1$

Now, from the intercept form of the line

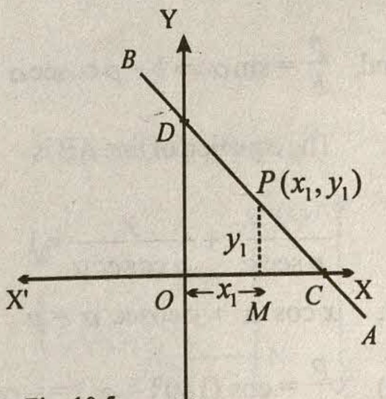


Fig. 10.5

$$\frac{x}{2x_1}+\frac{y}{2y_1}=1$$

or, $$\frac{x}{x_1}+\frac{y}{y_1}=2$$

which is the required equation of the line.

(e) PERPENDICULAR FORM (NORMAL FORM)

We now derive the equation of a line when p be the length of perpendicular from the origin on the line and α, the angle which this perpendicular makes with the positive direction of x-axis are given.

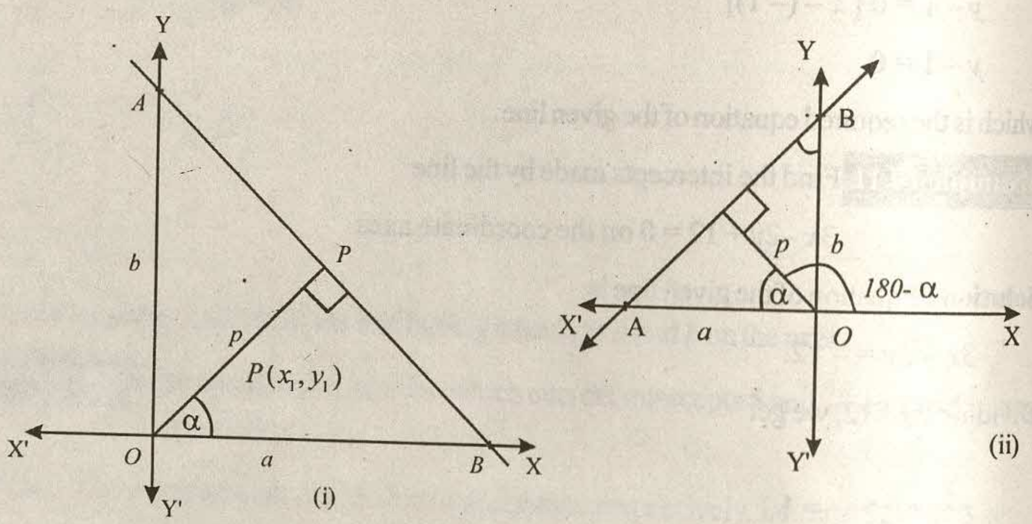


Fig. 10.6

(i) Let AB be the given line cutting off intercepts a and b on x–*axis* and y–*axis* respectively. Let OP be perpendicular from origin O on AB and $\angle POB = \alpha$ (See Fig. 10.6 (i))

$\therefore \quad \frac{p}{a} = \cos\alpha \Rightarrow a = p\sec\alpha$

and, $\frac{p}{b} = \sin\alpha \Rightarrow b = p\,\mathrm{cosec}\,\alpha$

$\therefore$ The equation of line AB is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\mathrm{cosec}\,\alpha} = 1$$

or, $x\cos\alpha + y\,\mathrm{cosec}\,\alpha = p$

(ii) $\frac{p}{a} = \cos(180° - \alpha) = -\cos\alpha$ [From Fig. 10.6 (ii)]

$\Rightarrow \quad a = -p\sec\alpha$

similary, $b = p\,\mathrm{cosec}\,\alpha$

$\therefore$ The equation of the line AB is $\frac{x}{-a} + \frac{y}{b} = 1$

$x\cos\alpha + y\sin\alpha = p$

Note : 1. p is the length of perpendicular from the origin on the line and in always taken to be positive.

2. α is the angle between positive direction of x-axis and the line perpendicular from the origin to the given line.

Example 10.13 Determine the equation of the line with $\alpha = 135°$ and perpendicular distance $p = \sqrt{2}$ from the origin.

Solution : From the standard equation of the line in normal form have

$$x \cos 135° + y \sin 135° = \sqrt{2}$$

or, $-\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = \sqrt{2}$

or, $-x + y - 2 = 0$

or, $x - y + 2 = 0$

which is the required equation of the straight line.

Example 10.14 Find the equation of the line whose perpendicular distance from the origin is 6 units and the perpendicular from the origin to line makes an angle of $30°$ with the positive direction of x-axis.

Solution : Here $\alpha = 30°, p = 6$

$\therefore$ The equation of the line is

$$x \cos 30° + y \sin 30° = 6$$

or, $x\left(\dfrac{\sqrt{3}}{2}\right) + y\left(\dfrac{1}{2}\right) = 6$

or, $\sqrt{3}\, x + y = 12$

(f) PARAMETRIC FORM

We now want to find the equation of a line through a given point $Q(x_1, y_1)$ which makes an angle α with the positive direction of $x-$axis in the form

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} = r$$

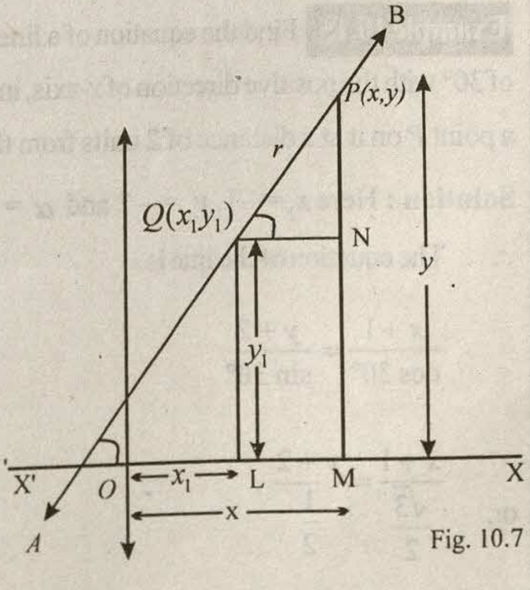


Fig. 10.7

where r is the distance of any point P (x, y) on the line from Q (x_1, y_1).

Let AB be a line passing through a given point $Q(x_1, y_1)$ making an angle α with the positive direction of x-axis.

Let P(x, y) be any point on the line such that $QP = r$ (say)

Draw QL, PM perpendiculars to x-axis and draw $QN \perp PM$.

From right angled Δ PNQ

$$\cos\alpha = \frac{QN}{QP}, \sin\alpha = \frac{PN}{QP}$$

But $\quad \mathrm{QN} = \mathrm{LM} = \mathrm{OM} - \mathrm{OL} = \mathrm{x} - \mathrm{x}_1$

and $\quad \mathrm{PN} = \mathrm{PM} - \mathrm{MN} = \mathrm{PM} - \mathrm{QL} = \mathrm{y} - \mathrm{y}_1$

$$\therefore \quad \cos\alpha = \frac{x - x_1}{r}, \sin\alpha = \frac{y - y_1}{r}$$

or, $\quad \dfrac{x - x_1}{\cos\alpha} = \dfrac{y - y_1}{\sin\alpha} = r$, which is the required equation of the line

Note : 1. The co-ordinates of any point on this line can be written as $(x_1 + r\cos\alpha, y_1 + r\sin\alpha)$. Clearly coordinates of the point depend on the value of r. This variable r is called parameter.

2. The equation $x = x_1 + r\cos\alpha$, $y = y_1 + r\sin\alpha$ are called the parametric equations of the line.

3. The value of 'r' is positive for all points lying on one side of the given point and negative for all points lying on the other side of the given point.

Example 10.15 Find the equation of a line passing through A $(-1, -2)$ and making an angle of 30° with the positive direction of x-axis, in the parametric form. Also find the coordinates of a point P on it at a distance of 2 units from the point A.

Solution : Here $x_1 = -1, y_1 = -2$ and $\alpha = 30°$

$\therefore \quad$ The equation of the line is

$$\frac{x+1}{\cos 30°} = \frac{y+2}{\sin 30°}$$

or, $\quad \dfrac{x+1}{\dfrac{\sqrt{3}}{2}} = \dfrac{y+2}{\dfrac{1}{2}}$

Since AP = 2, $r = 2$

$$\therefore \quad \frac{x+1}{\frac{\sqrt{3}}{2}} = \frac{y+2}{\frac{1}{2}} = 2$$

$$\Rightarrow \quad x = \sqrt{3} - 1, y = -1$$

Thus the coordinates of the point P are $\left(\sqrt{3} - 1, -1\right)$

Example 10.16 Find the distance of the point (1, 2) from the line $2x - 3y + 9 = 0$ measured along a line making an angle of 45° with $x-$axis.

Solution : The equation of any line through A (1, 2) making an angle of 45° with x-axis is

$$\frac{x-1}{\cos 45^\circ} = \frac{y-2}{\sin 45^\circ} = r$$

$$\frac{x-1}{\frac{1}{\sqrt{2}}} = \frac{y-2}{\frac{1}{\sqrt{2}}} = r$$

Any point on it is $\left(1 + \frac{r}{\sqrt{2}}, 2 + \frac{r}{\sqrt{2}}\right)$

It lies on the given line $2x - 3y + 9 = 0$

if $\quad 2\left(1 + \frac{r}{\sqrt{2}}\right) - 3\left(2 + \frac{r}{\sqrt{2}}\right) + 9 = 0$

or, $\quad 2 - 6 + 9 - \frac{r}{\sqrt{2}} = 0$

or, $\quad r = 5\sqrt{2}$

Hence the required distance is $5\sqrt{2}$.

CHECK YOUR PROGRESS 10.2

1. (a) Find the equation of a line with slope 2 and $y-$intercept is equal to -2.

   (b) Determine the slope and the intercepts made by the line on the axes whose equation is $4x + 3y = 6$.

2. Find the equation of the line cutting off an interecept $\frac{1}{\sqrt{3}}$ on negative direction of axis of y and inclined at 120° to the positive direction of x-axis.

3. Find the slope and y-intercept of the line whose equation is $3x - 6y = 12$.

4. Determine the equation of the line passing through the point $(-7, 4)$ and having the slope $-\frac{3}{7}$.

5. Determine the equation of the line passing through the point $(1, 2)$ which makes equal angles with the two axes.

6. Find the equation of the line passing through $(2, 3)$ and parallel to the line joining the points $(2, -2)$ and $(6, 4)$.

7. (a) Determine the equation of the line through $(3, -4)$ and $(-4, 3)$.

   (b) Find the equation of the diagonals of the rectangle ABCD whose vertices are A $(3, 2)$, B $(11, 8)$, C $(8, 12)$ and D $(0, 6)$.

8. Find the equation of the medians of a triangle whose vertices are $(2, 0)$, $(0, 2)$ and $(4, 6)$.

9. Find the equation of the line which cuts off intercepts of length 3 units and 2 units on x-axis and y-axis respectively.

10. Find the equation of a line such that the segment between the coordinate axes has its mid point at the point $(1, 3)$

11. Find the equation of a line which passes through the point $(3, -2)$ and cuts off positive intercepts on x and y axes in the ratio of $4 : 3$.

12. Determine the equation of the line whose perpendicular from the origin is of length 2 units and makes an angle of 45° with the positwe direction of x-axis.

13. If p is the length of the perpendicular segment from the origin, on the line whose intercept on the axes are a and b, then show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

14. Find the equation of a line passing through A $(2, 1)$ and making an angle 45° with the positive direction of x-axis in parametric form. Also find the coordinates of a point P on it at a distance of 1 unit from the point A.

10. 3 GENERAL EQUATION OF FIRST DEGREE

You know that *a* linear equation in two variables *x* and *y* is given by $Ax+By+C=0$ ...(1)

In order to understand its graphical representation, we need to take the following thres cases.

Case-1: (When both A and B are equal to zero)

In this case C is automaticaly zero and the equation does not exist.

Case-2: (When $A=0$ and $B\neq 0$)

In this case the equation (1) becomes $By+C=0$.

or $y=-\frac{C}{B}$ and is satisfied by all points lying on a line which is parallel to *x*-axis and the *y*-coordinate of every point on the line is $-\frac{C}{B}$. Hence this is the equation of a straight line. The case where $B=0$ and $A\neq 0$ can be treated similarly.

Case-3: (When $A\neq 0$ and $B\neq 0$)

We can solve the equation (1) for *y* and obtain.

$$y=-\frac{A}{B}x\not-\frac{C}{B}$$

Clearly, this represents a straight line with slope $-\frac{A}{B}$ and $y-$intercept equal to $-\frac{C}{B}$.

10.3.1 CONVERSION OF GENERAL EQUATION OF A LINE INTO VARIOUS FORMS

If we are given the general equation of a line, in the form $Ax+By+C=0$, we will see how this can be converted into various forms studied before.

10.3.2 CONVERSION INTO SLOPE-INTERCEPT FORM

We are given a first degree equation in *x* and *y* as $Ax+By+C=O$

Are you able to find slope and *y*-intercept ?

Yes, indeed, if we are able to put the general equation in slope-intercept form. For this purpose, let us re-arrange the given equation as.

$$Ax+By+C=0 \text{ as}$$

$$By=-Ax-C$$

or $$y=-\frac{A}{B}x-\frac{C}{B} \quad \text{(Provided } B\neq 0\text{)}$$

which is the required form. Hence, the slope $=-\frac{A}{B}$, $y-$intercept $=-\frac{C}{B}$.

Example 10.17 Reduce the equation $x + 7y - 4 = 0$ to the slope – intercept form.

Solution : The given equation is

$$x + 7y - 4 = 0$$

or $$7y = -x + 4$$

or $$y = -\frac{1}{7}x + \frac{4}{7}$$

which is the converted form of the given equation in slope-intercept form

Example 10.18 Find the slope and y intercept of the line $x + 4y - 3 = 0$.

Solution : The given equation is

$$x + 4y - 3 = 0$$

or $$4y = -x + 3$$

or $$y = -\frac{1}{4}x + \frac{3}{4}$$

Comparing it with slope-intercept form, we have

$$\text{slope} = -\frac{1}{4}, y - \text{intercept} = \frac{3}{4}.$$

10.3.3 CONVERSION INTO INTERCEPT FORM

Suppose the given first degree equation in x and y is

$$Ax + By + C = 0. \qquad \text{...(i)}$$

In order to convert (i) in intercept form, we re arrange it as

$$Ax + By = -C \text{ or } \frac{Ax}{-C} + \frac{By}{-C} = 1$$

or $$\frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \quad (\text{Provided } A \neq 0 \text{ and } B \neq 0)$$

which is the requied converted form. It may be noted that intercept on x – axis $= \frac{-C}{A}$ and

intercept on y – axis $= \frac{-C}{B}$

Example 10.19 Reduce $3x + 5y = 7$ into the intercept form and find its intercepts on the axes.

Solution : The given equation is

$$3x + 5y = 7$$

or, $$\frac{3}{7}x + \frac{5}{7}y = 1$$

or, $$\frac{x}{\frac{7}{3}} + \frac{y}{\frac{7}{5}} = 1$$

$\therefore$ The x– intercept $= \dfrac{7}{3}$ and, y – intercept $= \dfrac{7}{5}$

Example 10.20 Find x and y-intercepts for the line $3x - 2y = 5$.

Solution : The given equation is

$$3x - 2y = 5$$

or, $$\frac{3}{5}x - \frac{2}{5}y = 1$$

or, $$\frac{x}{\frac{5}{3}} + \frac{y}{\frac{-5}{2}} = 1$$

Thus, the required x – intercept $= \dfrac{5}{3}$

and y-intercept $= -\dfrac{5}{2}$

10.3.4 CONVERSION INTO PERPENDICULAR FORM

Let the general first degree equation in x and y be

$$Ax + By + C = 0 \qquad \text{... (i)}$$

We will convert this general equation in perpendicular form. For this purpose let us re-write the given equation (i) as $Ax + By = - C$

Multiplying both sides of the above equation by λ, we have

$$\lambda Ax + \lambda By = - \lambda C \qquad \text{... (ii)}$$

Let us choose λ such that $(\lambda A)^2 + (\lambda B)^2 = 1$

or $\lambda = \dfrac{1}{\sqrt{(A^2+B^2)}}$ (Taking positive sign)

Substituting this value of λ in (ii), we have

$$\frac{Ax}{\sqrt{(A^2+B^2)}} + \frac{By}{\sqrt{(A^2+B^2)}} = -\frac{C}{\sqrt{(A^2+B^2)}} \qquad \text{...(iii)}$$

This is required conversion of (i) in perpendicular form. Two cases arise according as C is negative or positive.

(i) If $C < 0$, the equation (ii) is the required form.

(ii) If $C > 0$, the R. H. S. of the equation of (iii) is negative.

$\therefore$ We shall multiply both sides of the equation of (iii) by -1.

$\therefore$ The required form will be

$$-\frac{Ax}{\sqrt{(A^2+B^2)}} - \frac{By}{\sqrt{(A^2+B^2)}} = \frac{C}{\sqrt{(A^2+B^2)}}$$

Thus, length of perpendicular from the origin $= \dfrac{|C|}{\sqrt{(A^2+B^2)}}$

Inclination of the perpendicular with the positwe direction of x-ax is

is given by $\cos\theta = \mp \dfrac{A}{\sqrt{A^2+B^2}}$

or $\sin\theta = \left(\mp \dfrac{B}{\sqrt{(A^2+B^2)}}\right)$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$. If $C = 0$, the line passes through the origin and there is no perpendicular from the origin on the line.

With the help of the above three cases, we are able to say that

> ***"The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously."***

Is the converse of the above statement true? ***The converse of the above statement is that every straight line can be expressed as a general equation of first degree in x and y.***

In this lesson we have studied about the various forms of equation of straight line. For example, let us take some of them as $y = mx + c$, $\frac{x}{a} + \frac{y}{b} = 1$ and $x \cos \alpha + y \sin \alpha = p$. Obviously, all are linear equations in x and y. We can re-arrange them as $y - mx - c = 0$, $bx + ay - ab = 0$ and $x \cos \alpha + y \sin \alpha - p = 0$ respectively. Clearly, these equations are nothing but a different arrangement of general equation of first degree in x and y. Thus, we have established that

> ***"Every straight line can be expressed as a general equation of first degree in x and y".***

Example 10.21 Reduce the equation $x + \sqrt{3}\, y + 7 = 0$ into perpendicular form.

Solution : The equation of given line is $x + \sqrt{3}y + 7 = 0$ ... (i)

Comparing (i) with general equation of straight line, we have

$$A = 1 \text{ and } B = \sqrt{3}$$

$$\therefore \quad \sqrt{A^2 + B^2} = 2$$

Dividing equation (i) by 2, we have

$$\frac{x}{2} + \frac{\sqrt{3}}{2} y + \frac{7}{2} = 0$$

or $$\left(-\frac{1}{2}\right)x + \left(-\frac{\sqrt{3}}{2}\right)y - \frac{7}{2} = 0$$

or $$x \cos \frac{4\pi}{3} + y \sin \frac{4\pi}{3} = \frac{7}{2}$$

($\cos \theta$ and $\sin \theta$ being both negative in the third quadrant, value of θ will lie in the third quadrant).

This is the representation of the given line in perpendicular form.

Example 10.22 Find the perpendicular distance from the origin on the line $\sqrt{3}\, x - y + 2 = 0$. Also, find the inclination of the perpendicular from the origin.

Solution : The given equation is $\sqrt{3}\, x - y + 2 = 0$

Dividing both sides by $\sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, we have

$$\frac{\sqrt{3}}{2}x-\frac{1}{2}y+1=0$$

or, $$\frac{\sqrt{3}}{2}x-\frac{1}{2}y=-1$$

Multiplying both sides by –1, we have

$$-\frac{\sqrt{3}}{2}x+\frac{1}{2}y=1$$

or, $x\cos\frac{5\pi}{6}+y\sin\frac{5\pi}{6}=1$ (cos θ is –ve in second quadrant and sin θ is +ve in second quadrant, so value of θ lies in the second quadrant).

Thus, inclination of the perpendicular from the origin is 150° and its length is equal to 1.

Example 10.23 Find the equation of a line which passess through the point (3,1) and bisects the portion of the line $3x+4y=12$ intercepted between coordinate axes.

Solution : First we find the intercepts on coordinate axes cut off by the line whose equation is from. $3x+4y=12$

or $$\frac{3x}{12}+\frac{4y}{12}=1$$

or $$\frac{x}{4}+\frac{y}{3}=1$$

Hence, intercepts on *x*-axis and *y*-axis are 4 and 3 respectively.

Thus, the coordinates of the points where the line meets the coordinate axes are *A* (4, 0) and *B* (0, 3).

∴ Mid – point of *AB* is $\left(2,\frac{3}{2}\right)$.

Hence the equation of the line through (3, 1) and $\left(2,\frac{3}{2}\right)$ is

$$y-1=\frac{\frac{3}{2}-1}{2-3}(x-3)$$

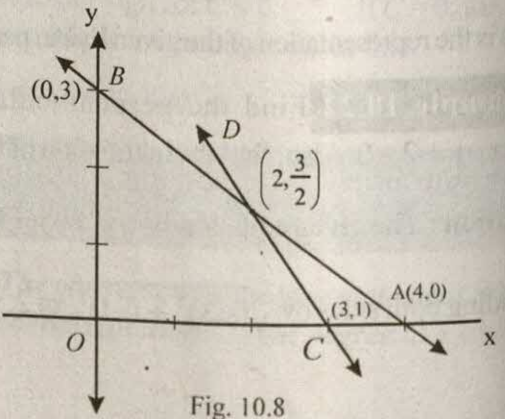


Fig. 10.8

or $\quad y-1=-\frac{1}{2}(x-3)$

or $\quad 2(y-1)+(x-3)=0$

or $\quad 2y-2+x-3=0$

or $\quad x+2y-5=0$

Example 10.24 Prove that the line through (8, 7) and (6, 9) cuts off equal intercepts on coordinate axes.

Solution : The equation of the line passing through (8, 7) and (6, 9) is

$$y-7=\frac{9-7}{6-8}(x-8)$$

or $\quad y-7=-(x-8)$

or $\quad x+y=15$

or $\quad \frac{x}{15}+\frac{y}{15}=1$

Hence, intercepts on both axes are 15 each.

Example 10.25 Find the ratio in which the line joining (–5, 1) and (1, –3) divides the join of (3, 4) and (7, 8).

Solution : The equation of the line joining C (–5, 1) and D (1, –3) is

$$y-1=\frac{-3-1}{1+5}(x+5)$$

or $\quad y-1=-\frac{4}{6}(x+5)$

or $\quad 3y-3=-2x-10$

or $\quad 2x+3y+7=0 \qquad \text{... (i)}$

Let line (i) divide the join of A (3, 4) and B (7, 8) at the point P.

If the required ratio is λ : 1 in which line (i) divides the join of A (3, 4) and B (7, 8), then the coordinates of P are

$$\left(\frac{7\lambda+3}{\lambda+1},\frac{8\lambda+4}{\lambda+1}\right)$$

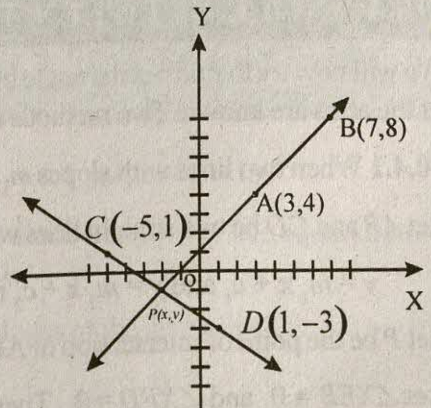


Fig. 10.9

Since P lies on the line (i), we have

$$2\left(\frac{7\lambda+3}{\lambda+1}\right)+3\left(\frac{8\lambda+4}{\lambda+1}\right)+7=0$$

$$\Rightarrow \quad 14\lambda+6+24\lambda+12+7\lambda+7=0$$

$$\Rightarrow \quad 45\lambda+25=0 \Rightarrow \lambda=-\frac{5}{9}$$

Hence, the line joining $(-5, 1)$ and $(1, -3)$ divides the join of $(3, 4)$ and $(7, 8)$ externally in the ratio $5:9$.

CHECK YOUR PROGRESS 10.3

1. Under what condition, the general equation $Ax + By + C = 0$ of first degree in x and y represents a line?
2. Reduce the equation $2x + 5y + 3 = 0$ to the slope intercept form.
3. Find the x and y intercepts for the following lines :

   (a) $y = mx + c$ (b) $3y = 3x + 8$ (c) $3x - 2y + 12 = 0$
4. Find the length of the line segment AB intercepted by the straight line $3x - 2y + 12 = 0$ between the two axes.
5. Reduce the equation $x \cos \alpha + y \sin \alpha = p$ to the intercept form of the equation and also find the intercepts on the axes.
6. Reduce the following equations into normal form.

   (a) $3x - 4y + 10 = 0$ (b) $3x - 4y = 0$
7. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is nearer from the origin?

We will here develop a formula for finding angle between two given lines

10.4 ANGLE BETWEEN TWO LINES

We will now try to find out the angle between the arms of the divider or clock when equation of the arms are known. Two methods are discussed below.

10.4.1 When two lines with slopes m_1 and m_2 are given

Let AB and CD be two straight lines whose equations are

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2 \text{ respectively.}$$

Let P be the point of intersection of AB and CD.

Let $\angle XEB = \theta_1$ and $\angle XFD = \theta_2$. Then

$\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$

Let θ be the angle $\angle EPF$ between AB and CD.

Now $\theta_1 = \theta_2 + \theta$

or $\quad \theta = \theta_1 - \theta_2$

$\therefore \quad \tan\theta = \tan(\theta_1 - \theta_2)$

or $\quad \tan\theta = \dfrac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$

Hence, $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

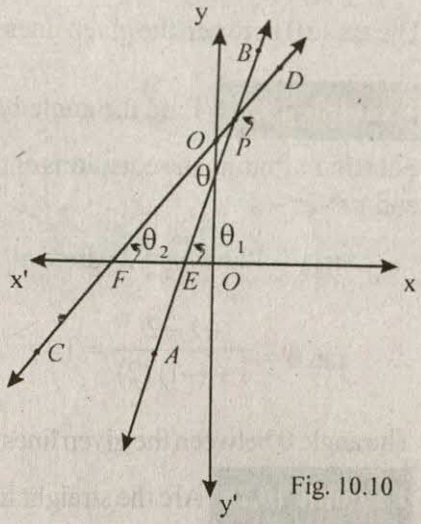


Fig. 10.10

which gives tangent of the angle θ between two given lines in terms of their slopes.

From this result we can find the angle between any two given lines when their slopes are given.

Note : (i) *If* $\dfrac{m_1 - m_2}{1 + m_1 m_2} > 0$, then angle between lines is **acute.**

(ii) *If* $\dfrac{m_1 - m_2}{1 + m_1 m_2} < 0$, then angle between lines is **obtuse.**

(iii) *If* $\dfrac{m_1 - m_2}{1 + m_1 m_2} = 0$,

i.e., $m_1 = m_2$, then lines are either ***coincident or parallel.***

i.e, Two lines are parallel if they differ only by a constant term.

(iv) If the denominator of $\dfrac{m_1 - m_2}{1 + m_1 m_2}$ is zero.

i.e, $m_1 m_2 = -1$, then lines are ***perpendicular.***

Example 10.26 Find the angle between the lines $7x - y = 1$ and $6x - y = 11$.

Solution : Let θ be the angle betwen the lines $7x - y = 1$ and $6x - y = 11$. The equations of these lines can be written as $y = 7x - 1$ and $y = 6x - 11$.

Here, $m_1 = 7$, $m_2 = 6$. where m_1 and m_2 are respective slopes of the given lines.

$\therefore \quad \tan\theta = \dfrac{7-6}{1+42} = \dfrac{1}{43}$

The angle θ between the given lines is given by $\tan\theta = \frac{1}{43}$

Example 10.27 Find the angle between the lines $x + y + 1 = 0$ and $2x - y - 1 = 0$.

Solution : Putting the equations of the given lines in slope intercept form, we get $y = -x - 1$ and $y = 2x - 1$

$\therefore$ $m_1 = -1$ and $m_2 = 2$ where m_1, and m_2 are respectively slopes of the given lines.

$$\therefore \quad \tan\theta = \frac{-1-2}{1+(-1)(2)} = 3$$

The angle θ between the given lines is given by $\tan\theta = 3$

Example 10.28 Are the straight lines $y = 3x - 5$ and $3x + 4 = y$, parallel or perpendicular?

Solution : For the first line $y = 3x - 5$, the slope $m_1 = 3$ and for the second line $3x + 4 = y$, the slope $m_2 = 3$. Since the slopes of both the lines are same for both the lines, hence the lines are parallel.

Example 10.29 Are the straight lines $y = 3x$ and $y = -\frac{1}{3}x$, parallel or perpendicular?

Solution : The slope m_1 of the line $y = 3x$ is 3 and slope m_2 of the second line $y = -\frac{x}{3}$ is $-\frac{1}{3}$.

Since, $m_1 m_2 = -1$, the lines are perpendicular.

10.4.2 When the Equation of two lines are in general form

Let the general equations of two given straight lines be given by

$$A_1 x + B_1 y + C_1 = 0 \qquad \text{... (i)}$$

and $$A_2 x + B_2 y + C_2 = 0 \qquad \text{...(ii)}$$

Equations (i) and (ii) can be written into their slope intercept from as

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1} \quad \text{and} \quad y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2}$$

Let m_1 and m_2 be their respective slopes and using the formula, we get.

$$\text{i.e.,} \quad \tan\theta = \frac{A_2 B_1 - A_1 B_2}{A_1 A_2 + B_1 B_2}$$

Thus, by using this formula we can calculate the angle between two lines when they are in general form.

Note : (i) If $A_2 B_1 - A_1 B_2 = 0$

or $A_1 B_2 - A_2 B_1 = 0$

i.e. $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$, then the given lines are parallel.

(ii) If $A_1 A_2 + B_1 B_2 = 0$, then the given lines are perpendicular.

Example 10.30 Are the straight lines $x - 7y + 12 = 0$ and $x - 7y + 6 = 0$, parallel?

Solution : Here $A = 1$, $B_1 = -7$; $A_2 = 1$ and $B_2 = -7$

Since $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ 1 & -7 \end{vmatrix} = 0$

Example 10.31 Determine whether the straight lines $2x + 5y - 8 = 0$ and $5x - 2y - 3 = 0$ are parallel or perpendicular ?

Here $A_1 = 2, B_1 = 5$

$A_2 = 5$ and $B_2 = -2$

Solution : Here $A_1 A_2 + B_1 B_2 = (2) \times (5) + (5) \times (-2) = 0$.

Hence the given lines are perpendicular.

CHECK YOUR PROGRESS 10.4

1. What is the condition for given lines $y = m_1 x + c_1$ and $Ax + By + C = 0$

   (a) to be parallel? (b) to be perpendicular?

2. Find the angle between the lines $4x + y = 3$ and $\frac{x}{2} + y = \frac{4}{7}$.

3. (a) Write the condition that the two lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ are

   (i) parallel.

   (ii) perpendicular.

   (b) Are the straight lines $x - 3y = 7$ and $2x - 6y - 16 = 0$, parallel?

   (c) Are the straight lines $x = y + 1$ and $x = -y + 1$ perpendicular?

10.5 DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

In this section, we shall discuss the concept of finding the distance of a given point from a given line or lines.

Let $P(x_1, y_1)$ be the given point and l be the line $Ax + By + C = 0$.

Let the line l intersect x axis and y axis R and Q respectively.

Draw $PM \perp l$ and let $PM = d$.

Let the coordinates of M be (x_2, y_2)

$$d = \sqrt{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}} \qquad \text{...(i)}$$

$\because$ M lies on l

$\therefore$ $Ax_2 + By_2 + C = 0$

or $C = -(Ax_2 + By_2)$ ...(ii)

The coordinates of R and Q are $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ respectively.

The slope of $QR = \dfrac{0 + \frac{C}{B}}{-\frac{C}{A} - 0} = -\dfrac{A}{B}$ and,

the slope of $PM = \dfrac{y_2 - y_1}{x_2 - x_1}$

As $PM \perp QR \Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} \times \left(-\dfrac{A}{B}\right) = -1.$

or $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{B}{A}$ ...(iii)

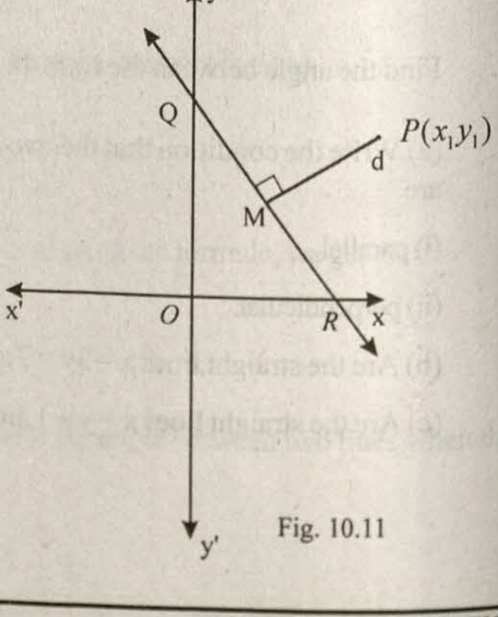


Fig. 10.11

From (iii) $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{\sqrt{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}}}{\sqrt{\left(A^2 + B^2\right)}}$ ...(iv)

(Using properties of Ratio and Proportion)

Also $\dfrac{x_1 - x_2}{A} = \dfrac{y - y_2}{B} = \dfrac{A\,(x_1 - x_2) + B\,(y_1 - y_2)}{A^2 + B^2.}$ ...(v)

From (iv) and (v), we get

$$\frac{\sqrt{\left\{(x_1 - x_2)^2\,(y_1 - y_2)^2\right\}}}{\sqrt{\left(A^2 + B^2\right)}} = \frac{A\,(x_1 - x_2) + B\,(y_1 - y_2)}{A^2 + B^2}$$

or $\dfrac{d}{\sqrt{A^2 + B^2}} + \dfrac{Ax_1 + By_1 - (Ax_2 + By_2}{A^2 + B^2}$ [Using (i)]

or $\dfrac{Ax_1 + By_1 + C}{\sqrt{\left(A^2 + B^2\right)}}$ [Using (ii)]

Since the distance is always positive, we can write

$$d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{\left(A^2 + B^2\right)}}\right|$$

Note : The perpendicular distance of the origin (0, 0) from Ax + By + C = 0 is

$$\frac{A(0) + B(0) + C}{\sqrt{\left(A^2 + B^2\right)}} = \frac{C}{\sqrt{\left(A^2 + B^2\right)}}$$

Example 10.32 Find the points on the x-axis whose perpendicular distance from the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is a.

Solution : Let $(x_1, 0)$ be any point on x-axis.

Equation of the given line is $bx + ay - ab = 0$. The perpendicular distance of the point $(x_1, 0)$ from the given line is

$$a = \pm \frac{bx_1 + a.0 - ab}{\sqrt{(a^2 + b^2)}}$$

$$\therefore \quad x_1 = \frac{a}{b}\left\{b \pm \sqrt{(a^2 + b^2)}\right\}$$

Thus, the point on x-axis is $\left(\frac{a}{b}\left(b \pm \sqrt{a^2 + b^2}\right), 0\right)$

CHECK YOUR PROGRESS 10.5

1. Find the perpendicular distance of the point (2, 3) from $3x + 2y + 4 = 0$.
2. Find the points on the axis of y whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is b.
3. Find the points on the axis of y whose perpendicular distance from the straight line $4x + 3y = 12$ is 4.
4. Find the perpendicular distance of the origin from $3x + 7y + 14 = 0$
5. What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{4} + \frac{y}{4} = 1$ is 3?

10.6 EQUATION OF PARALLEL (OR PERPENDICULAR) LINES

Till now, we have developed methods to find out whether the given lines are prallel or perpendicular. In this section, we shall try to find, the equation of a line which is parallel or perpendicular to a given line.

10.6.1 EQUATION OF A STRAIGHT LINE PARALLEL TO THE GIVEN LINE

$$Ax + By + c = 0$$

Let $A_1x + B_1y + C_1 = 0$ ...(i)

be any line parallel to the given line

$$Ax + By + C = 0$$

The condition for parallelism of (i) and (ii) is ...(ii)

$$\frac{A_1}{A} = \frac{B_1}{B} = K_1 \quad \text{(say)}$$

$\Rightarrow \quad A_1 = AK_1, B_1 = BK_1$

, with these values of A_1 and B_1, (i) gives

$$AK_1x + BK_1y + C_1 = 0$$

or $$Ax + By + \frac{C_1}{K_1} = 0$$

or $$Ax + By + K = 0, \qquad \text{where } K = \frac{C_1}{K_1} \qquad \text{... (iii)}$$

This is a line parallel to the given line. From equations (ii) and (iii) we observe that

(i) coefficients of x and y are same

(ii) constants are different, and are to evaluated from given conditions.

Example 10.33 Find equation of the straight line, which passes through the point (1, 2) and which is parallel to the straight line $2x + 3y + 6 = 0$.

Solution : Equation of any straight line parallel to the given equation can be written if we put

(i) the coefficients of x and y as same as in the given equation.

(ii) constant to be different from the given equation, which is to be evaluated under given condition.

Thus, the required equation of the line will be

$2x + 3y + K = 0$ for some constant K

Since it passes through the point (1, 2) hence

$$2 \times 1 + 3 \times 2 + K = 0$$

or $K = -8$

$\therefore$ Required equation of the line is $2x + 3y = 8$.

10.7 STRAIGHT LINE PERPENDICULAR TO THE GIVEN LINE

$$Ax + By + C = 0$$

Let $A_1x + B_1y + C_1 = 0$

be any line perpendicular to the given line ... (i)

$$Ax + By + C = 0$$

Condition for perpendicularity of lines (i) and (ii) is ... (ii)

$$AA_1 + BB_1 = 0$$

$$\Rightarrow \quad \frac{A_1}{B} = -\frac{B_1}{A} = K_1 \quad \text{(say)}$$

$\Rightarrow \quad A_1 = BK_1$ and $B_1 = -AK_1$

With these values of A_1 and B_1, (i) gives

$$Bx - Ay + \frac{C_1}{K_1} = 0 = 0$$

or $\quad Bx - Ay + K = 0$ where $K = \dfrac{C_1}{K_1}$ ... (iii)

Hence, the line (iii) is perpendicular to the given line (ii)

We observe that in order to get a line perpendicular to the given line we have to follow the following procedure :

(i) Interchange the coefficients of x and y

(ii) Change the sign of one of them.

(iii) Change the Constant term to a new constant K (say), and evaluate it from given condition.

Example 10.34 Find the equation of the line which passes through the point $(1, 2)$ and is perpendicular to the line $2x+3y+6=0$.

Solution : Following the procedure given above, we get the equation of line perpendicular to the given equation as $3x-2y+K=0$ ...(i)

(i) passes through the point $(1, 2)$, hence

$3 \times 1 - 2 \times 2 + K = 0$ or $K = 1$

$\therefore$ Required equation of the straight line is $3x-2y+1=0$.

Example 10.35 Find the equation of the line which passes through the point (x_2, y_2) and is perpendicular to the straight line $yy_1 = 2a\,(x+x_1)$.

Solution : The given straight line is $yy_1 - 2ax - 2ax_1 = 0$ ...(i)

Any straight line perpendicular to (i) is $2ay + xy_1 + C = 0$

This passes through the point (x_2, y_2)

$\therefore \quad 2ay_2 + x_2 y_1 + C = 0$

$\Rightarrow \quad C = -2ay_2 - x_2 y_1$

$\therefore$ Required equation of the spraight line is

$$2a\,(y - y_2) + y_1\,(x - x_2) = 0$$

CHECK YOUR PROGRESS 10.6

1. Find the equation of the straight line which passes through the point $(0, -2)$ and is parallel to the straight line $3x + y = 2$.

2. Find the equation of the straight line which passes through the point $(-1, 0)$ and is parallel to the straight line $y = 2x + 3$.

3. Find the equation of the straight line which passes through the point $(0, -3)$ and is perpendicular to the straight line $x + y + 1 = 0$.

4. Find the equation of the line which passes through the point $(0, 0)$ and is perpendicular to the straight line $x + y = 3$.

5. Find the equation of the straight line which passes through the point $(2, -3)$ and is perpendicular to the given straight line $2a(x + 2) + 3y = 0$.

6. Find the equation of the line which has x - intercept -8 and is perpendicular to the line $3x + 4y - 17 = 0$.

7. Find the equation of the line whose y-intercept is 2 and is parallel to the line $2x - 3y + 7 = 0$.

8. Prove that the equation of a straight line passing throngh $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

10.8 APPLICATION OF COORDINATE GEOMETRY

Let us take some examples to show how coordinate geometry can be gainfully used to prove some geometrical results.

Example 10.36 Prove that the diagonals of a rectangle are equal.

Solution : Let us take origin as one vertex of the rectangle without any loss of generality and take one side along *x*–axis of length *a* and another side of length b along *y*–axis, as shown in Fig. 10.12. Then the coordinates of the points A, B and C are $(a, 0)$, (a, b) and $(0, b)$ respectively

$$\text{Length of diagonal AC} = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$$

$$\text{Length of diagonal OB} = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$$

$\therefore \quad AC = OB$

i.e., the diagonals of the rectangle OA BC are equal. Thus, in general, we can say that the length of the diagonals of a rectangle are equal.

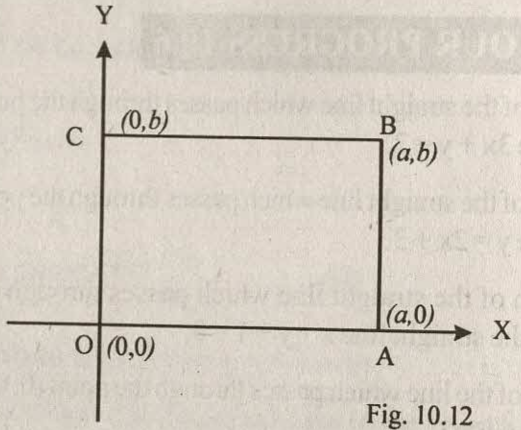


Fig. 10.12

Example 10.37 Prove that angle in a semi circle is a right angle.

Solution : Let O (0,0) be the centre of the circle and $(-a,0)$ and $(a,0)$ be the end- points of one of its diameter. Let the circle intersects the y-axis at $(0, a)$ [see Fig. 10.13]

$$\therefore \quad \text{Slope of BQ} = \frac{a-0}{0-a} = 1 = m_1$$

$$\text{Slope of AQ} = \frac{0-a}{a-0} = -1 = m_2$$

As $m_1 m_2 = -1 \Rightarrow$ AQ is perpendicular to BQ

$\therefore$ BQ A is a right angle.

Again, let $P(a\cos\theta, a\sin\theta)$ be a general point on the circle.

$$\therefore \quad \text{Slope of PB} = \frac{a\sin\theta}{a\cos\theta + a} = \frac{\sin\theta}{1+\cos\theta} = m_1 \text{ (say)}$$

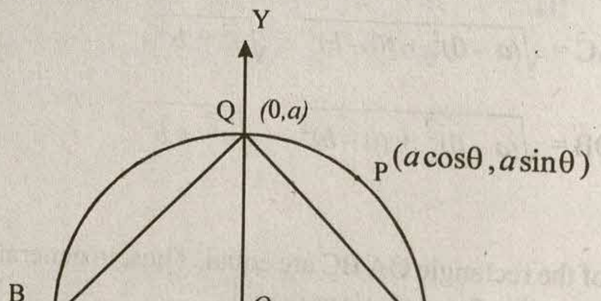


Fig. 10.13

Slope of PA $= \dfrac{0 - a\sin\theta}{a - a\cos\theta} = \dfrac{-\sin\theta}{1-\cos\theta} = m_2$ (say)

and $m_1 . m_2 = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)\left(-\dfrac{\sin\theta}{1-\cos\theta}\right) = -\dfrac{\sin^2\theta}{1-\cos^2\theta} = -\dfrac{\sin^2\theta}{1-\cos^2\theta} = -1$

$\therefore$ PA is perpendicular to PB.

∠BPA is a right angle.

Example 10.38 Medians drawn from two vertices to equal sides of an isosceles triangle are equal.

Solution : Let OAB be an equilateral triangle with vertices (0,0), (a,0) and $\left(\dfrac{a}{2}, a\right)$

Let C and D be the mid-points of AB and OB respectively.

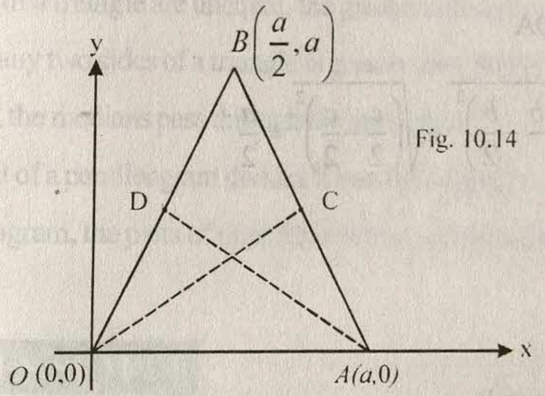


Fig. 10.14

$\therefore$ C has coordinates $\left(\dfrac{3a}{4}, \dfrac{a}{2}\right)$ and

D has coordinates $\left(\dfrac{a}{4}, \dfrac{a}{2}\right)$

$\therefore$ Length of OC $= \sqrt{\left(\dfrac{3a}{4}\right)^2 + \left(\dfrac{a}{2}\right)^2} = \sqrt{\dfrac{9a^2}{16} + \dfrac{a^2}{4}} = \dfrac{a}{4}\sqrt{13}$

and Length of AD $= \sqrt{\left(a - \dfrac{a}{4}\right)^2 + \left(0 - \dfrac{a}{2}\right)^2} = \dfrac{a}{4}\sqrt{13}$

$\therefore$ Length of OC = length of AD

Hence the proof.

Example 10.39 The line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half of it.

Solution : Let the coordinates of the vertices be O (0, 0), A (a, 0) and B (b,.c). Let C and D be the mid-points of OB and AB respectively

$\therefore$ The coordinates of C are $\left(\frac{b}{2}, \frac{c}{2}\right)$

The coordinates of D are $\left(\frac{a+b}{2}, \frac{c}{2}\right)$

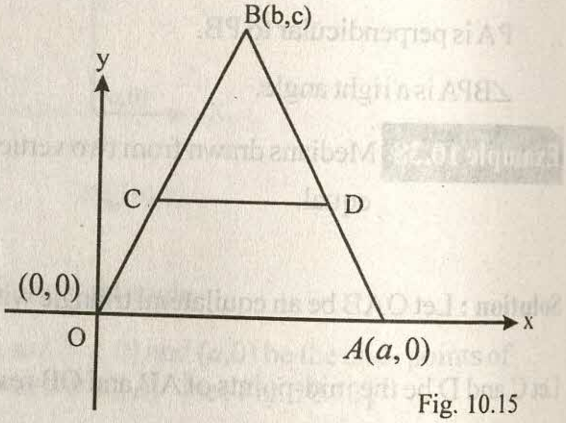


Fig. 10.15

$$\therefore \quad \text{Slope of CD} = \frac{\frac{c}{2}-\frac{c}{2}}{\frac{a+b}{2}-\frac{b}{2}} = 0$$

Slope of OA, i.e, $x-$axis $= 0$

$\therefore$ CD is parallel to OA

$$\text{Length of CD} = \sqrt{\left(\frac{a+b}{2}-\frac{b}{2}\right)^2} + \sqrt{\left(\frac{c}{2}-\frac{c}{2}\right)^2} = \frac{a}{2}$$

Length of OA $= a$

$$\therefore \quad \text{CD} = \frac{1}{2}\ \text{OA}$$

which proves the above result.

Example 10.40 If the diagonals of a quadrilateral are perpendicular and bisect each other, then the quadrilateral is a rhombus.

Solution : Let the diagonals be represented along OX and OY as shown in Fig. 10.16 and suppose that the lengths of their diagonals be $2a$ and $2b$ respectively.

$$\text{OA} = \text{OC} = a \text{ and OD} = \text{OB} = b$$

$$\text{AD} = \sqrt{OA^2 + OD^2} = \sqrt{a^2 + b^2}$$

$$\text{AB} = \sqrt{OB^2 + OA^2} = \sqrt{a^2 + b^2}$$

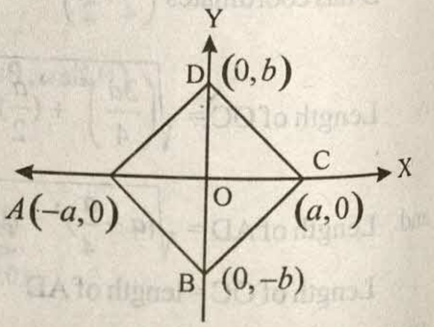


Fig. 10.16

Similarly $BC = \sqrt{a^2 + b^2} = CD$

As $AB = BC = CD = DA$

$\therefore$ A BCD is a rhumbus.

CHECK YOUR PROGRESS 10.7

Using coordinate geometry, prove the following geometrical results:

1. If two medians of a triangle are equal, then the triangle is isosceles.
2. The diogonals of a square are equal and perpendicular to each other.
3. If the diagonals of a parallelogram are equal, then the paralle logram is a rectangle.
4. If D is the mid-point of base BC of Δ ABC, then $AB^2 + AC^2 = 2(AD^2 + BD^2)$
5. In a triangle, if a line is drawn parallel to one side, it divides the other two sides proportionally.
6. If two sides of a triangle are unequal, the greater side has a greater angle opposite to it.
7. The sum of any two sides of a triangle is greater than the third side.
8. In a triangle, the medians pass throngh the same point.
9. The diagonal of a paralleogram divides it into two triangles of equal area.
10. In a parallelogram, the pairs of opposite sides are of equal length.

LET US SUM UP

- The equation of a line parallel to y-axis is $x = a$ and parallel to x-axis is $y = b$.
- The equation of the line which cuts off intercept c on y-axis and having slope m is $y = mx + c$
- The equation of the line passing through $A(x_1, y_1)$ and having the slope m is $y - y_1 = m(x - x_1)$
- The equation of the line passing through two points $A(x, y_1)$ and $B(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- The equation of the line which cuts off intercepts a and b on $x-$axis and $y-$axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- The equation of the line in normal or perpendicular form is $x\cos\alpha + y\sin\alpha = p$

where p is the length of perpendicular from the origin to the line and α is the angle which this perpendicular makes with the positive direction of the x-axis.

- The equation of a line in the parametric form is $\dfrac{x-x_1}{\cos\alpha}=\dfrac{y-y_1}{\sin\alpha}=r$

where r is the distance of any point $P(x,y)$ on the line from a given point $A(x_1,y_1)$ on the given line and α is the angle which the line makes with the positive direction of the x-axis.

- The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously.

- From general equation $Ax+By+C=0$ we can evaluate the following :

(i) Slope of the line $=-\dfrac{A}{B}$ (ii) x-intercept $=-\dfrac{C}{A}$ (iii) y-intercept $=-\dfrac{C}{B}$

(iv) Length of perpendicular from the origin to the line $=\dfrac{|C|}{\sqrt{(A^2+B^2)}}$

(v) Inclination of the perpendicular from the origin is given by

$$\cos\alpha=\frac{\mp A}{\sqrt{(A^2+B^2)}}\ ;\ \sin\alpha=\frac{\mp B}{\sqrt{(A^2+B^2)}}$$

where the upper sign is taken for $C>0$ and the lower sign for $C<0$; but if $C=0$ then either only the upper sign or only the lower sign are taken.

- If θ be the angle between two lines with slopes m_1 m_2 then $\tan\theta=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

(i) Lines are parallel if $m_1=m_2$. (ii) Lines are perpendicular if $m_1\,m_2=-1$.
(B) Angle θ between two lines of general form is:

$$\tan\theta=\frac{A_2B_1-A_1B_2}{A_1A_2+B_1B_2}$$

(i) Lines are parallel if $A_1\,B_2-A_2\,B_1=0$ or $\begin{vmatrix}A_1 & B_1\\ A_2 & B_2\end{vmatrix}=0$
(ii) Lines are perpendicular if $A_1A_2+B_1\,B_2=0$.

- Distance of a given point (x_1,y_1) from a given line $Ax+By+C=0$ is $d=\left|\dfrac{Ax_1+By_1+C}{\sqrt{(A^2+B^2)}}\right|$
- Equation of a line parallel to the line $Ax+By+C=0$ is $Ax+By+k=0$
- Equation of a line perpendicular to the line $Ax+By+C=0$ is $Bx-Ay+k=0$

Notes

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equation of the straight line whose y-intercept is -3 and which is:

   (a) parallel to the line joining the points $(-2,3)$ and $(4,-5)$.

   (b) perpendicular to the line joining the points $(0,-5)$ and $(-1,3)$.

2. Find the equation of the line passing through the point $(4,-5)$ and

   (a) parallel to the line joining the points $(3,7)$ and $(-2,4)$.

   (b) perpendicular to the line joining the points $(-1,2)$ and $(4,6)$.

3. Show that the points $(a,0)$, $(0,b)$ and $(3a,-2b)$ are collinear. Also find the equation of the line containing them.

4. A $(1,4)$, B $(2,-3)$ and C $(-1,-2)$ are the vertices of triangle ABC. Find

   (a) the equation of the median through A.

   (b) the equation of the altitude througle A.

   (c) the right bisector of the side BC.

5. A straight line is drawn through point A $(2,1)$ making an angle of $\frac{\pi}{6}$ with the positive direction of x-axis. Find the equation of the line.

6. A straight line passes through the point $(2,3)$ and is parallel to the line $2x+3y+7=0$. Find its equation.

7. Find the equation of the line having a and b as x-intercept and y-intercepts respectively.

8. Find the angle between the lines $y=(2-\sqrt{3})x+5$ and $y=(2+\sqrt{3})x-d$.

9. Find the angle between the lines $2x+3y=4$ and $3x-2y=7$

10. Find the length of the per pendicular drawn from the point $(3,4)$ on the straight line $12(x+6)=5(y-2)$.

11. Find the length of the perpendicula from $(0,1)$ on $3x+4y+5=0$.

12 Find the distance between the lines

$$2x+3y=4 \text{ and } 4x+6y=20$$

Notes

13. Find the length of the perpendicular drawn from the point $(-3, -4)$ on the line $4x - 3y = 7$.

14. Prove that the equation of the straight line which passes through the point $(a \cos^3 \theta, b \sin^3 \theta)$ and is perpendicular to $x \sec\theta + y \operatorname{cosec} \theta = a$ is $x \cos\theta - y \operatorname{cosec} \theta = a \cos 2\theta$

15. Prove the following geometrical results :

(a) The altitudes of a triangle are concurrent.

(b) The medians of an equaliateral triangle are equal.

(c) The area of an equalateral triangle of side a is $\dfrac{\sqrt{3}\, a^2}{4}$

(d) If G is the centroid of Δ ABC, and O is any point, then $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + CG^2 + 3OG^2$.

(e) The perpendicular drawn from the centre of a circle to a chord bisects the chord.

ANSWERS

CHECK YOUR PROGRESS 10.1

1. Straight line 2. (a) 1 (b) −1 (c) 1

3. 7 4. 20 5. (a) $y=-4$ (b) $x=-3$

6. $x=5$ 7. $y+7=0$

CHECK YOUR PROGRESS 10.2

1. (a) $y=2x-2$ (b) Slope $=\frac{-4}{3}$, y − intercept = 2

2. $\sqrt{3}y=-3x-1$ 3. Slope $=\frac{1}{2}$, y-intercept $=-2$

4. $3x+7y=7$ 5. $y=x+1; x+y-3=0$ 6. $3x-2y=0$

7. (a) $x+y=-1$ (b) Equation of the diagonal AC $=2x-y-4=0$

Equation of the diagonal BD $=2x-11y+66=0$

8. $x-2=0, x-3y+6=9$ and $5x-3y-2=0$

9. $2x+3y=6$ 10. $3x+y=6$ 11. $3x+4y=1$ 12. $x+y=2\sqrt{2}$

14. $\dfrac{x-2}{\frac{1}{\sqrt{2}}}=\dfrac{y-1}{\frac{1}{\sqrt{2}}}$ and the co-ordinates of the point are $\left(2+\frac{1}{\sqrt{2}},1+\frac{1}{\sqrt{2}}\right)$

CHECK YOUR PROGRESS 10.3

1. A and B are not both simltaneously zero 2. $y=\frac{-2}{5}x-\frac{3}{5}$

3. (a) x −intercept $=\frac{-c}{m}$; y-intercept = c (b) x-intercept $=\frac{-8}{3}$; y-intercept $=\frac{8}{3}$

(c) x-intercept $=-4$; y-intercept $=6$

4. $2\sqrt{13}$ units 5. $\dfrac{x}{p\sec\alpha}+\dfrac{x}{p\,\mathrm{cosec}\,\alpha}=1$

6. (a) $\frac{-3}{5}x+\frac{4}{5}y-2=0$ (b) $\frac{-3}{5}x+\frac{4}{5}y=0$

7. The first line is nearer from the origin.

CHECK YOUR PROGRESS 10.4

1. (a) $m_1=\frac{-A}{B}$ (b) $Am_1=B$

2. $\theta=\tan^{-1}\left(\frac{7}{6}\right)$

3. (a) (i) $\frac{A_1}{B_1}=\frac{A_2}{B_2}$ (ii) $A_1A_2+B_1B_2=0$

(b) Parallel (c) Perpendicular.

CHECK YOUR PROGRESS 10.5

1. $d=\frac{16}{\sqrt{13}}$ 2. $\left(0,\frac{b}{a}(a\pm\sqrt{a^2+b^2})\right)$ 3. $\left(0,\frac{32}{3}\right)$

4. $\frac{14}{\sqrt{58}}$ 5. $\left(\frac{3}{4}(4\pm 5),0\right)$

CHECK YOUR PROGRESS 10.6

1. $3x+y+2=0$ 2. $y=2x+2$ 3. $x-y=3$ 4. $y=x$

5. $3x-2ay=6(a-1)$ 6. $4x-3y+32=0$ 7. $2x-3y+6=0$

TERMINAL EXERCISE

1. (a) $4x+3y+9=0$ (b) $x-8y-24=0$

2. (a) $3x-5y-37=0$ (b) $5x-8y-60=0$

4. (a) $13x-y-9=0$ (b) $3x-y+1=0$

(c) $3x-y-4=0$

5. $x-\sqrt{3}y=2-\sqrt{3}$ 6. $2x+3y+13=0$

7. $bx+ay=ab$ 8. $\frac{\pi}{2}$ 9. $\frac{\pi}{2}$ 10. $\frac{98}{13}$

11. $\frac{9}{5}$ 12. $\frac{6}{\sqrt{13}}$ 13. $\frac{7}{5}$

11

CIRCLES

Notice the path in which the tip of the hand of a watch moves. (see Fig. 11.1)

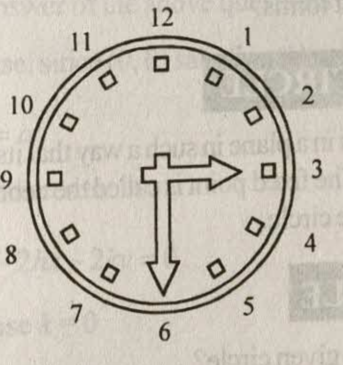


Fig. 11.1

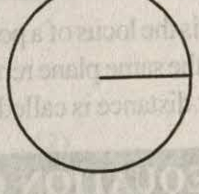


Fig. 11.2

Again, notice the curve traced out when a nail is fixed at a point and a thread of certain length is tied to it in such a way that it can rotate about it, and on the other end of the thread a pencil is tied. Then move the pencil around the fixed nail keeping the thread in a stretched position (See Fig 11.2)

Certainly, the curves traced out in the above examples are of the same shape and this type of curve is known as a ***circle***.

The distance between the tip of the pencil and the point, where the nail is fixed is known as the ***radius*** of the circle.

We shall discuss about the curve traced out in the above examples in more details.

OBJECTIVES

After studying this lesson, you will be able to :

- derive and find the equation of a circle with a given centre and radius;

- state the conditions under which the general equation of second degree in two variables represents a circle;
- find the centre and radius of a circle whose equation is given in general form;
- find the equation of a circle passing through :

  (*i*) three non-collinear points (*ii*) two given points and touching any of the axes;
- derive the equation of a circle in the diameter form;
- find the equation of a circle when the end points of any of its diameter are given; and
- find the parametric representation of a circle with given centre and radius.

EXPECTED BACKGROUND KNOWLEDGE

- Terms and concepts connected with circle.
- Distance between two points with given coordinates.
- Equation of a straight line in different forms.

11.1 DEFINITION OF THE CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the same plane remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

11.2 EQUATION OF A CIRCLE

Can we find a mathematical equation for a given circle?

Let us try to find the equation of a circle under various given conditions.

11.2.1 WHEN COORDINATES OF THE CENTRE AND RADIUS ARE GIVEN

Let C be the centre and a be they radius of the circle. Coordinates of the centre are given to be (h, k), say.

Take any point $P(x, y)$ on the circle and draw perpendiculars CM and PN on *x-axis*. Again, draw CL perpendicular to PN.

We have

$$CL = MN = ON - OM = x - h$$

and $PL = PN - LN = PN - CM = y - k$

In the right angled triangle CLP, $CL^2 + PL^2 = CP^2$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2 \qquad \text{...(1)}$$

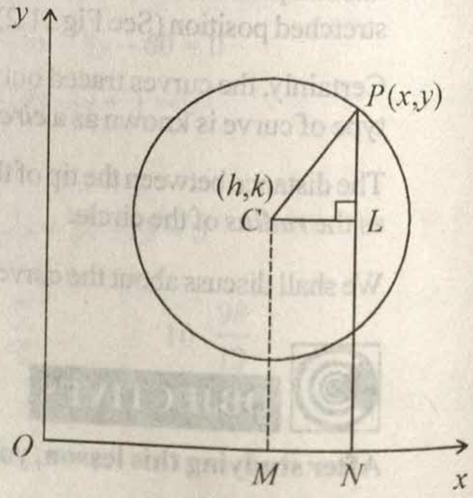


Fig. 11.3

This is the required equation of the circle under given conditions. This form of the circle is known as ***standard form*** of the circle.

Conversely, if (x, y) is any point in the plane satisfying (1), then it is at a distance 'a' from (h, k). So it is on the circle.

What happens when the

(*i*) circle passes through the origin?

(*ii*) circle does not pass through origin and the centre lies on the x-axis?

(*iii*) circle passes through origin and the x-axis is a diameter?

(*iv*) centre of the circle is origin?

(*v*) circle touches the x-axis?

(*vi*) circle touches the y-axis?

(*vii*) circle touches both the axes?

We shall try to find the answer of the above questions one by one.

(*i*) In this case, since (0, 0) satisfies (1), we get

$$h^2 + k^2 = a^2$$

Hence the equation (1) reduces to

$$x^2 + y^2 - 2hx - 2ky = 0 \qquad ...(2)$$

(*ii*) In this case $k = 0$

Hence the equation (1) reduces to

$$(x - h)^2 + y^2 = a^2 \qquad ...(3)$$

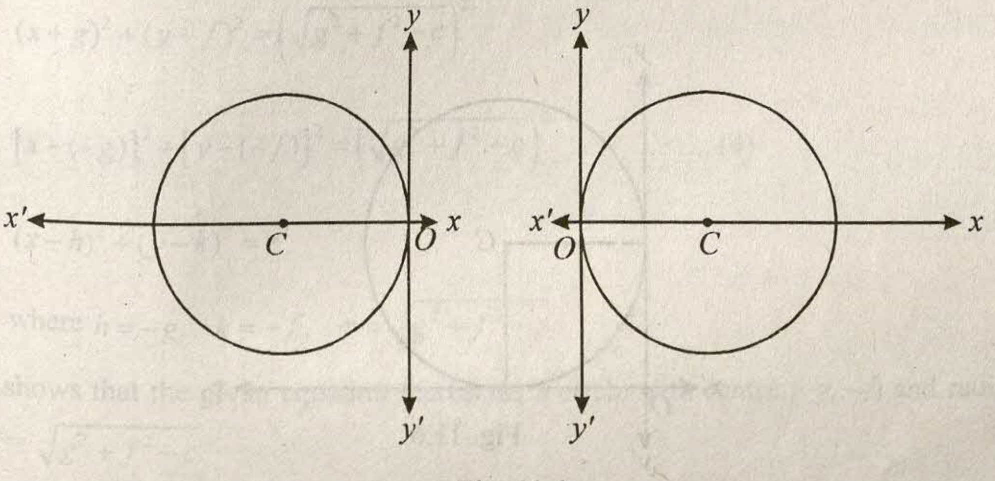


Fig. 11.4

(*iii*) In this case $k = 0$ and $h = \pm a$ (see Fig. 11.4)

Hence the equation (1) reduces to $x^2 + y^2 \pm 2ax = 0$ ...(4)

(*iv*) In this case $h = 0 = k$

Hence the equation (1) reduces to $x^2 + y^2 = a^2$ ...(5)

(*v*) In this case $k = a$ (see Fig. 11.5)

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ay + h^2 = 0$ ...(6)

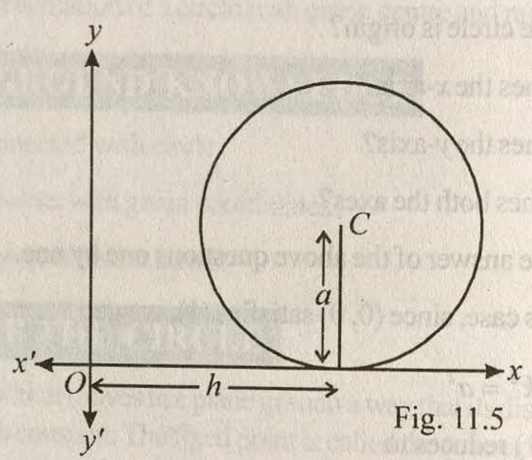


Fig. 11.5

(*vi*) In this case $h = a$

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ky + k^2 = 0$ ...(7)

(*vii*) In this case $h = k = a$. (See Fig. 11.6)

Hence the equation (1) reduces to $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ ...(8)

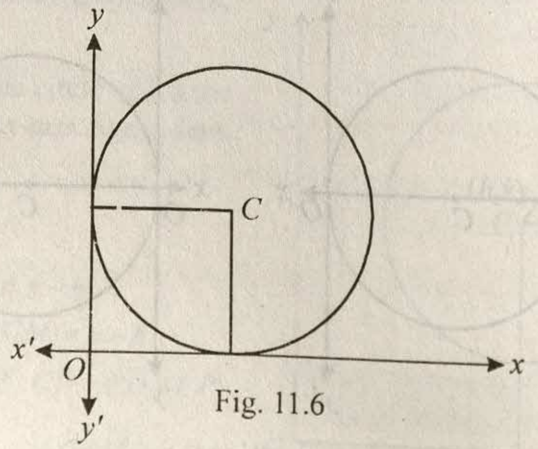


Fig. 11.6

Example 11.1 Find the equation of the circle whose centre is $(3, -4)$ and radius is 6.

Solution : Comparing the terms given in equation (1), we have

$h = 3, k = -4$ and $a = 6$.

$$\therefore \quad (x-3)^2 + (y+4)^2 = 6^2$$

or
$$x^2 + y^2 - 6x + 8y - 11 = 0$$

Example 11.2 Find the centre and radius of the circle given by $(x+1)^2 + (y-1)^2 = 4$.

Solution: Comparing the given equation with $(x-h)^2 + (y-k)^2 = a^2$ we find that

$-h = 1, -k = -1, a^2 = 4$

$\therefore \quad h = -1, k = 1, a = 2.$

So the given circle has its centre $(-1, 1)$ and radius 2.

11.3 GENERAL EQUATION OF THE CIRCLE IN SECOND DEGREE IN TWO VARIABLES

The standard equation of a circle with centre (h, k) and radius r is given by

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{... (1)}$$

or
$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \qquad \text{... (2)}$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (3)}$$

$$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$\Rightarrow \quad [x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \qquad \text{... (4)}$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$$

where $h = -g, \quad k = -f, \quad r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$

11.3.1 CONDITONS UNDER WHICH THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES REPRESENTS A CIRCLE

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) It is a second degree equation in x, y in which coefficients of the terms involving x^2 and y^2 are equal.

(ii) It contains no term involving xy

Note : In solving problems, we keep the coefficients of x^2 and y^2 unity.

Example 11.3 Find the centre and radius of the circle

$$45x^2 + 45y^2 - 60x + 36y + 19 = 0$$

Solution : Given equation can be written on dividing by 45 as

$$x^2 + y^2 - \frac{4}{3}x + \frac{4}{5}y + \frac{19}{45} = 0$$

Comparing it with the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ we get}$$

$$g = -\frac{2}{3}, f = \frac{2}{5} \text{ and } c = \frac{19}{45}$$

Thus, the centre is $\left(\frac{2}{3}, -\frac{2}{5}\right)$ and radius is $\sqrt{g^2 + f^2 - c} = \frac{\sqrt{41}}{15}$

Example 11.4 Find the equation of the circle which passes through the points (1, 0), (0, – 6) and (3, 4).

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(1)}$$

Since the circle passes through three given points so they will satisfy the equation (1). Hence

$$1 + 2g + c = 0 \qquad \cdots(2)$$

and
$$36 - 12f + c = 0 \qquad \cdots(3)$$

$$25 + 6g + 8f + c = 0 \qquad \cdots(4)$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g + 12f = 35$$

and $\quad 6g + 20f = 11$

Solving these equations for *g* and *f*, we get $g=-\frac{71}{4},\ f.=\frac{47}{8}$

Substituting *g* in (2), we get $c=\frac{69}{2}$

and substituting *g*, *f* and *c* in (1), the required equation of the circle is

$$4x^2+4y^2-142x+47y+138=0$$

Exmaple 11. 5 Find the equation of the circles which touches the axis of *x* and passes through the points (1,–2) and (3,–4).

Solution : Since the circle touches the *x*-axis, put $k=a$ in the standard form (See result 6) of the equation of the circle, we have

$$x^2+y^2-2hx-2ay+h^2=0 \qquad \text{... (1)}$$

This circle passes through the point (1,–2)

$$\therefore \quad h^2-2h+4a+5=0 \qquad \text{... (2)}$$

Also, the circle passes through the point (3, – 4)

$$\therefore \quad h^2-6h+8a+25=0 \qquad \text{...(3)}$$

Eliminationg '*a*' from (2) and (3), we get

$$\Rightarrow \quad h^2+2h-15=0$$

$$h=3 \ or \ h=-5.$$

From (3) the corresponding values of *a* are – 2 and –10 respectively. On substituting the values of *h* and *a* in (1) we get

$$x^2+y^2-6x+4y+9=0 \qquad \text{... (4)}$$

and $$x^2+y^2+10x+20y+25=0 \qquad \text{... (5)}$$

(4) and (5) represent the required equations.

11.4 EQUATION OF A CIRCLE WHEN END POINTS OF ONE OF ITS DIAMETERS ARE GIVEN.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given end points of the diameter *AB* (See Fig. 11.7)

Let $P(x, y)$ be any point on the circle drawn on *AB* as diameter. Join *AP* and *BP*. Since the angle in a semi-circle is a right angle.

$\therefore \quad AP \perp BP$

$\therefore \quad$ (slope of AP) × (slope of BP) $= -1$.

Now, the slope of $AP = \dfrac{y-y_1}{x-x_1}$

and the slope of $BP = \dfrac{y-y_2}{x-x_2}$

$$\therefore \quad \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

or $\quad (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \qquad ....(1)$

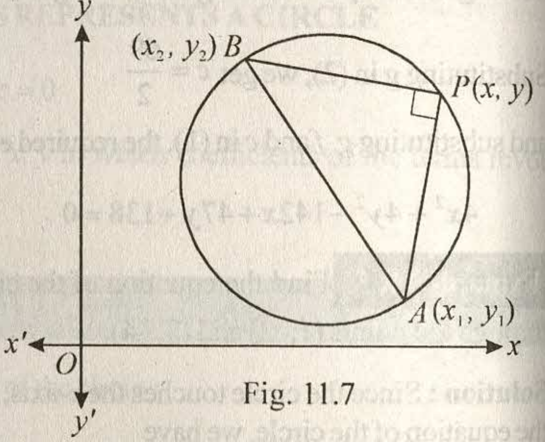


Fig. 11.7

Since (1) is true for every point on the circle, and for no other point in the plane:

$\therefore \quad$ (1) represents the equation of the circle in ***diameter form.***

Example 11.6 Find the equation of the circle described on the line joining the origin and the point (2,– 4) as diameter.

Solutoon: Here $x_1 = 0,\ y_1 = 0;\ x_2 = 2, y_2 = -4$

Using the Equation (1) required the equation of the circle is,

$$(x-0)(x-2)+(y-0)[y-(-4)]=0$$

or $\quad x^2 - 2x + y^2 + 4y = 0$

or $\quad x^2 + y^2 - 2x + 4y = 0$

Example 11.7 The equation of a chord of the circle $x^2 + y^2 - 2ax = 0$ is $y = mx$. Find the equation of the circle described on this chord as diameter.

Solution: The coordinates of the points of intersection of the circle and the given chord are,

$$(0, 0) \text{ and } \left(\frac{2a}{1+m^2}, \frac{2ma}{1+m^2}\right).$$

Now, for the required equation of the circle these points are the end points of the diameter, so by equation (1)

$$(x-0)\left(x-\frac{2a}{1+m^2}\right)+(y-0)\left(y-\frac{2ma}{1+m^2}\right)=0$$

or $$x^2+y^2-\frac{2a}{1+m^2}x-\frac{2ma}{1+m^2}y=0$$

or $$(1+m^2)x^2+(1+m^2)y^2-2ax-2may=0$$

This is the required equation of the circle.

11.5 EQUATION OF A CIRCLE WHEN RADIUS AND ITS INCLINATION ARE GIVEN (PARAMATRIC FORM)

In order to find the equation of the circle whose centre is the origin and whose radius is r. Let $P(x,\ y)$ be any point on the circle. Draw $PM \perp OX$

$\therefore$ $OM = x,\ MP = y$. Join OP.

Let $\angle XOP = \theta$ and $OP = r$

Now, $x = OM = r\cos\theta$

and $y = MP = r\sin\theta$

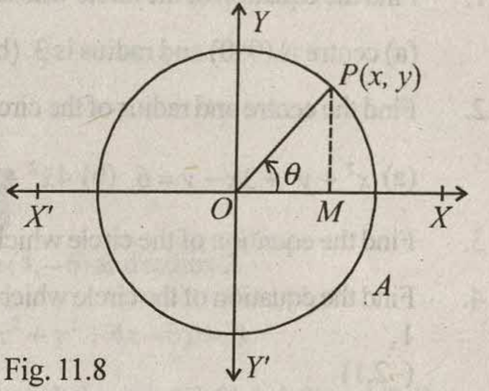


Fig. 11.8

Hence the two equations $x = r\cos\theta$ and $y = r\sin\theta$ taken together represent a circle. These are known as ***parametric form of the equations*** of the circle, where θ is a parameter.

> *Note : If the centre of the circle is at h and k, then the parametric form of the equation of the circle is* $x = h + r\cos\theta$ and $y = k + r\sin\theta$ where θ is a parametre which lies in the interval $[0, 2\pi]$.

Example 11.8 Find the parametric form of each of the following circles:

(i) $(x-1)^2+(y+2)^2=9$ (ii) $(x+2)(x-4)+(y-3)(y+1)=0$

Solution : (i) The equation of the circle is

$$(x-1)^2+(y-2)^2=3^2$$

Comparing it with the 'standard equation' of the circle, we get centre $(1,-2)$ and radius $= 3$ for the given circle.

$\therefore$ Parametric form of the equation of the circle is

$$x = 1+3\cos\theta;\ y = -2+3\sin\theta$$

(ii) Equation of the circle can be written as

$$x^2+y^2-2x-2y-11=0$$

Comparing it with the general equation of the circle and finding centre and radius, we get

Centre (1, 1) and radius = $\sqrt{13}$

$\therefore$ Parametric form of the equation of the circle is

$$x = 1 + \sqrt{13}\cos\theta,\ y = 1 + \sqrt{13}\sin\theta$$

CHECK YOUR PROGRESS 11.1

1. Find the equation of the circle whose

   (a) centre is (0, 0) and radius is 3. (b) centre is (– 2,3) and radius is 4.

2. Find the centre and radius of the circle

   (a) $x^2 + y^2 + 3x - y = 6$ (b) $4x^2 + 4y^2 - 2x + 3y - 6 = 0$

3. Find the equation of the circle which passes through the points (0, 2) (2, 0) and (0, 0).

4. Find the equation of the circle which touches the *y*-axis and passes through the points (–1, 2) and (–2,1)

5. Find the equation of the circle described on the line joining the points (2, 3) and (–2, 6) as diameter.

6. Find the parametric form of the equation of each of the following circles:

   (a) $(x+1)^2 + (y+1)^2 = 4$ (b) $4x^2 + 4y^2 + 2x + 2y - 3 = 0$

   (c) $(x-1)(x+1) + (y-1)(y+1) = 0$

LET US SUM UP

- **Standard form of the circle**

  $(x-h)^2 + (y-k)^2 = a^2$

  Centre is (*h*, *k*) and radius is *a*

- The general form of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

  Its centre: (–*g*, –*f*) and radius = $\sqrt{g^2 + f^2 - c}$

- **Diameter form of the circle**

  If the end points of a diameter are (x_1, y_1) and (x_2, y_2), then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Notes

- **Parametric form of the circle**

$x = a\cos\theta$, $y = a\sin\theta$ represents the parametric equation of a circle whose centre is at (0, 0) and radius $= a$

If the centre of the circle is at (h, k) then the parametric equation of the circle is

$x = h + a\cos\theta$ and $y = k + a\sin\theta$.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equation of a circle with centre (4, –6) and radius 7.
2. Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$
3. Find the equation of the circle passes through the point (1,0), (–1,0) and (0,1)
4. Find the parametric form of the equation of circles given below :

(a) $x^2 + y^2 = 3$ (b) $x^2 + y^2 - 4x + 6y = 12$

ANSWERS

CHECK YOUR PROGRESS 11.1

1. (a) $x^2+y^2=9$ (b) $x^2+y^2+4x-6y-3=0$

2. (a) $\left(-\frac{3}{2},1\right);\ \frac{\sqrt{37}}{2}$ (b) $\left(\frac{1}{4},-\frac{3}{8}\right);\ \frac{\sqrt{109}}{8}$

3. $x^2+y^2-2x-2y=0$ 4. $x^2+y^2+2x-2y+1=0$

5. $x^2+y^2-9y+14=0$

6. (a) $x=-1+2\cos\theta$ and $y=-1+2\sin\theta$.

   (b) $x=-\frac{1}{4}+\sqrt{\frac{7}{8}}\cos\theta \quad y=-\frac{1}{4}+\sqrt{\frac{7}{8}}\sin\theta$

   (c) $x=\sqrt{2}\cos\theta$ and $y=\sqrt{2}\sin\theta$.

TERMINAL EXERCISE

1. $x^2+y^2-8x+12y+3=0$ 2. Centre $(-2, 3)$; Radius $=\sqrt{13}$

3. $x^2+y^2=1$.

4. (a) $x=\sqrt{3}\cos\theta,\ \ y=\sqrt{3}\sin\theta$.

   (b) $x=2+5\cos\theta,\ \ y=-2+5\sin\theta$.

12

CONIC SECTIONS

While cutting a carrot you might have noticed different shapes shown by the edges of the cut. Analytically you may cut it in three different ways, namely

(*i*) Cut is parallel to the base (see Fig.12.1)

(*ii*) Cut is slanting but does not pass through the base (see Fig.12.2)

(*iii*) Cut is slanting and passes through the base (see Fig.12.3)

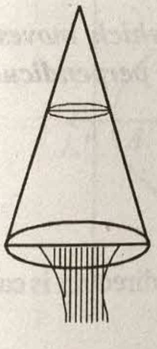


Fig. 12.1

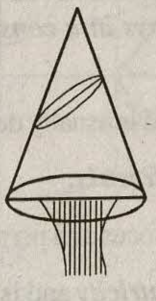


Fig. 12.2

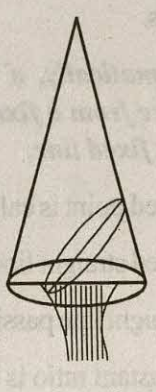


Fig. 12.3

The different ways of cutting, give us slices of different shapes.

In the first case, the slice cut represent a circle which we have studied in previous lesson.

In the second and third cases the slices cut represent different geometrical curves, which we shall study in this lesson.

OBJECTIVES

After studying this leson, you will be able to :

- recognise a circle, parabola and ellipse as sections of a cone;
- recognise the parabola and ellipse as certain loci;
- identify the concept of eccentricity, directrix, focus and vertex of a conic section;
- identify the standard equations of parabola and ellipse; and
- find the equation of a parabola given its directrix and focus.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of coordinate Geometry
- Various forms of equation of a straight line
- Equation of a circle in various forms

12.1 CONIC SECTION

In the introduction we have noticed the various shapes of the slice of the carrot. Since the carrot is conical in shape so the section formed are sections of a cone. They are therefore called conic sections.

Mathematically, a conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed line.

The fixed point is called the ***focus*** and is usually denoted by S.

The fixed straight line is called the ***Directrix.***

The straight line passing through the focus and perpendicular to the directrix is called the ***axis.***

The constant ratio is called the ***eccentricity*** and is denoted by e.

What happens when

(*i*) $e<1$ (*ii*) $e=1$ (*iii*) $e>1$

In these cases the conic section obtained are known as ellipse, parabola and hyperbola respectively.

In this lesson we shall study about ellipse and parabola only.

12.2 ELLIPSE

Recall the cutting of slices of a carrot. When we cut it obliquely, slanting without letting the knife pass through the base, what do we observe?

You might have come across such shapes when you cut a boiled egg vertically.

The slice thus obtained represents an ellipse. Let us define the ellipse mathematically as follows:

> ***"An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity".***

12.2.1 STANDARD EQUATION OF AN ELLIPSE

Let S be the focus, ZK be the directrix and P be a moving point. Draw SK perpendicular from S on the directrix. Let e be the eccentricity.

Divide SK internally and externally at A and A' (on KS produced) repectively in the ratio $e : 1$, as $e<1$.

$$SA = e.AK \qquad \cdots (1)$$

and $$SA' = e.A'K \qquad \cdots (2)$$

Since A and A' are points such that their distances from the focus bears a constant ratio e ($e < 1$) to their respective distances from the directrix and so they lie on the ellipse. These points are called vertices of the ellipse.

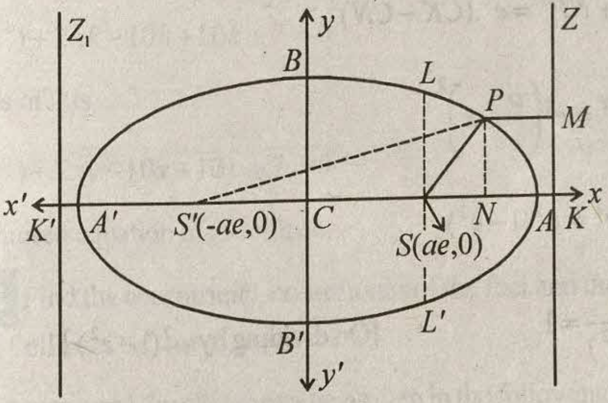


Fig. 12.4

Let AA' be equal to $2a$ and C be its mid point, i.e., $CA = CA' = a$

The point C is called the centre of the ellipse.

Adding (1) and (2), we have

$$SA + SA' = e.AK + e.A'K$$

or $$AA' = e(CK - CA + A'C + CK)$$

or $$2a = e.2CK$$

or $$CK = \frac{a}{e} \qquad \cdots (3)$$

Subtracting (1) from (2), we have

$$SA' - SA = e(A'K - AK)$$

or $$(SC + CA') - (CA - CS) = e.A'A$$

or $$2\,CS = e.2a$$

or $$CS = ae \qquad \cdots (4)$$

Let us choose C as origin, CAX as x-axis and CY, a line perpendicular to CX as y-axis.

$\therefore$ Coordinates of S are then $(ae, 0)$ and equation of the directrix is $x = \dfrac{a}{e}$

Let the coordinates of the moving point P be (x, y). Join SP, draw $PM \perp ZK$.

By definition $SP = e.PM$

or $$SP^2 = e^2 . PM^2$$

or $$SN^2 + NP^2 = e^2 .(NK)^2$$

or $$(CN - CS)^2 + NP^2 = e^2 .(CK - CN)^2$$

or $$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

or $$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \qquad \text{[On dividing by } a^2(1 - e^2)\text{]}$$

Putting $a^2(1 - e^2) = b^2$, we have the standard form of the ellipse as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Major axis : The line joining the two vertices A' and A, i.e., $A'A$ is called the major axis and its length is $2a$.

Minor axis : The line passing through the centre perpendicular to the major axis, i.e., BB' is called the minor axis and its length is $2b$.

Principal axis : The two axes together (major and minor) are called the principal axes of the ellipse.

Notes

Latus rectum : The length of the line segment LL' is called the latus rectum.

Equation of the directrices : $x = \pm\frac{a}{e}$

Eccentricity : e is given by $e^2 = 1 - \frac{b^2}{a^2}$

Example 12.1 Find the equation of the ellipse whose focus is (1,–1), eccentricity $e = \frac{1}{2}$ and the directrix is $x - y = 3$.

Solution : Let $P\ (h,k)$ be any point on the ellipse then by the definition, its distance from the focus = e. Its distance from directrix

or $\quad SP^2 = e^2.PM^2$

(M is the foot of the perpendicular drawn from P to the directrix).

or $\quad (h-1)^2 + (k+1)^2 = \frac{1}{4}\left(\frac{h-k-3}{\sqrt{1+1}}\right)^2$

or $\quad 7(h^2 + k^2) + 2hk - 10h + 10k + 7 = 0$

$\therefore$ The locus of P is

$$7(x^2 + y^2) + 2xy - 10x + 10y + 7 = 0$$

which is the required equation of the ellipse.

Example 12.2 Find the eccentricity, coordinates of the foci and the length of the axis of the ellipse $3x^2 + 4y^2 = 12$

Solution : The equation of the ellipse can be written in the following form

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

On comparing this equation with that of the standard equation of the ellipse, we have $a^2 = 4$ and $b^2 = 3$, then

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

(ii) coordinates of the foci are (1,0) and (–1,0)

[$\because$ The coordinate are ($\pm$ ae, 0)] (*iii*) Length of the major axes $= 2a = 2 \times 2 = 4$ and

length of the minor axis $= 2b = 2 \times \sqrt{3} = 2\sqrt{3}.$

CHECK YOUR PROGRESS 12.1

1. Find the equation of the ellipse referred to its centre (0,0)

   (a) whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$

   (b) whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

   (c) whose foci are the points (4,0) and (–4,0) and whose eccentricity is $\frac{1}{3}$.

2. Find the eccentricity of the ellipse, if its latus rectum be equal to one half its minor axis.

12.3 PARABOLA

Recall the cutting of slice of a carrot. When we cut obliquely and letting the knife pass through the base, what do we observe?

Also when a batsman hits the ball in air, have you ever noticed the path of the ball?

Is there any property common to the edge of the slice of the carrot and the path traced out by the ball in the example cited above?

Yes, the edge of such a slice and path of the ball have the same shape which is known as a parabola. Let us define parabola mathematically.

> ***"A parabola is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is equal to its distance from a fixed line in the plane."***

12.3.1 STANDARD EQUATION OF A PARABOLA

Let S be the fixed point and ZZ' be the directrix of the parabola. Draw SK perpendicular to ZZ'. Bisect SK at A.

Since $SA = AK$, by the definition of the parabola A lies on the parabola. A is called the vertex of the parabola.

Take A as origin, AX as the x-axis and AY perpendicular to AX through A as the y-axis.

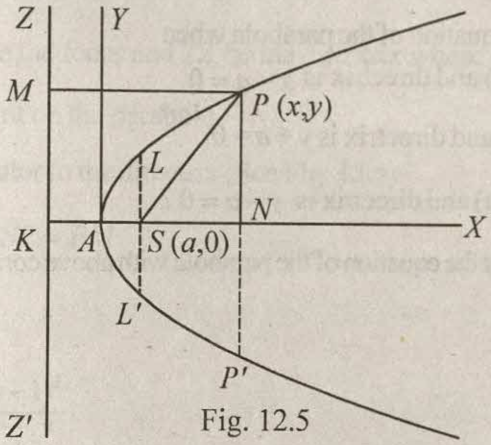


Fig. 12.5

Let $KS = 2a$ $\quad\quad \therefore AS = AK = a$

$\therefore$ The coordinates of A and S are $(0,0)$ and $(a,0)$ respectively.

Let $P(x,y)$ be any point on the parabola. Draw $PN \perp AS$ produced

$\therefore$ $AN = x$ and $NP = y$

Join SP and draw $PM \perp ZZ'$

$\therefore$ By definition of the parabola

$SP = PM$

or $SP^2 = PM^2$

or $(x-a)^2 + (y-0)^2 = (x+a)^2$

$[\because PM = NK = NA + AK = x + a]$

or $(x-a)^2 - (x+a)^2 = -y^2$

or $y^2 = 4ax$

which is the ***standard equation*** of the parabola.

Note : In this equation of the parabola

(*i*) Vertex is $(0,0)$

(*ii*) Focus is $(a,0)$

(*iii*) Equation of the axis is $y = 0$

(*iv*) Equation of the directrix is $x + a = 0$

(*v*) Latus rectum $= 4a$

12.3.2 OTHER FORMS OF THE PARABOLA

What will be the equation of the parabola when

(*i*) focus is $(-a,0)$ and directrix is $x-a=0$

(*ii*) focus is $(0,a)$ and directrix is $y+a=0$,

(*iii*) focus is $(0,-a)$ and directrix is $y-a=0$?

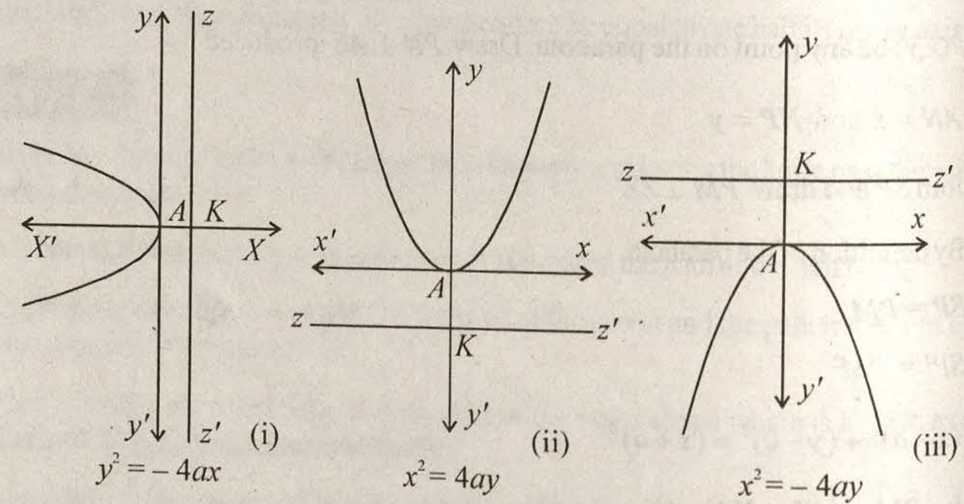
It can easily be shown that the equation of the parabola with above conditions takes the following forms:

(*i*) $y^2=-4ax$

(*ii*) $x^2=4ay$

(*iii*) $x^2=-4ay$

The figures are given below for the above equations of the parabolas.



(i) $y^2=-4ax$

(ii) $x^2=4ay$

(iii) $x^2=-4ay$

Fig. 12.6

Corresponding results of above forms of parabolas are as follows:

Forms	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Coordinates of vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Coordinates of focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Equation of directrix	$x=-a$	$x=a$	$y=-a$	$y=a$
Equation of the axis	$y=0$	$y=0$	$x=0$	$x=0$
Length of Latus rectum	$4a$	$4a$	$4a$	$4a$

Example 12.3 Find the equation of the parabola whose focus is the origin and whose directrix is the line $2x+y-1=0$.

Solution : Let $S(0,0)$ be the focus and ZZ' be the directrix whose equation is $2x+y-1=0$

Let $P(x,y)$ be any point on the parabola.

Let PM be perpendicular to the directrix (See Fig. 12.5)

$\therefore$ By definition $SP = PM$

or $SP^2 = PM^2$

or $$x^2+y^2=\frac{(2x+y-1)^2}{\left(\sqrt{2^2+1}\right)^2}$$

or $5x^2+5y^2=4x^2+y^2+1+4xy-2y-4x$

or $x^2+4y^2-4xy+2y+4x-1=0.$

Example 12.4 Find the equation of the parabola, whose focus is the point (2, 3) and whose directrix is the line $x-4y+3=0$.

Solution : Given focus is $S(2,3)$; and the equation of the directrix is $x-4y+3=0$.

$\therefore$ As in the above example

$$(x-2)^2+(y-3)^2=\left\{\frac{x-4y+3}{\sqrt{1^2+4^2}}\right\}^2$$

$\Rightarrow$ $16x^2+y^2+8xy-74x-78y+212=0$

CHECK YOUR PROGRESS 12.2

1. Find the equation of the parabola whose focus is (a, b) and whose directrix is $\frac{x}{a}+\frac{y}{b}=1$.

2. Find the equation of the parabola whose focus is (2,3) and whose directrix is $3x+4y=1$.

LET US SUM UP

- **Conic Section**

"A conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line".

(*i*) **Focus :** The fixed point is called the focus.

(*ii*) **Directrix :** The fixed straight line is called the directrix.

(*iii*) **Axis :** The straight line passing through the focus and pependicular to the directrix is called the axis.

(*iv*) **Eccentricity :** The constant ratio is called the eccentricity.

- (*v*) **Latus Rectum :** The double ordinate passing through the focus and parallel to the directrix is known as latus rectum. (In Fig.12.5 LSL' is the latus rectum).

- **Standard Equation of the Ellipse is :** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(*i*) Major axis $=2a$ (*ii*) Minor axis $=2b$

(*iii*) Equation of directrix is $x=\pm\frac{a}{e}$ (*iv*) Foci : $(\pm ae,0)$

(*v*) Eccentricity, i.e., e is given by $e^2=1-\frac{b^2}{a^2}$

- **Standard Equation of the Parabola is :** $y^2=4ax$

(*i*) Vertex is $(0,0)$ (*ii*) Focus is $(a,0)$

(*iii*) Axis of the parabola is $y=0$ (*iv*) Directrix of the parabola is $x+a=0$

(*v*) Latus rectum $=4a$.

- **OTHER FORMS OF THE PARABOLA ARE**

(*i*) $y^2=-4ax$ (concave to the left).

(*ii*) $x^2=4ay$ (concave upwards).

(*iii*) $x^2=-4ay$ (concave downwards).

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the equation of the ellipse in each of the following cases, when

   (a) focus is (0, 1), directrix is $x + y = 0$ and $e = \frac{1}{2}$.

   (b) focus is (–1,, 1), directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.

2. Find the coordinates of the foci and the eccentricity of each of the following ellipses:

   (a) $4x^2 + 9y^2 = 1$ (b) $25x^2 + 4y^2 = 100$

3. Find the equation of the parabola whose focus is $(-8, -2)$ and directrix is $y - 2x + 9 = 0$.

ANSWERS

CHECK YOUR PROGRESS 12.1

1. (a) $20x^2 + 36y^2 = 405$ (b) $x^2 + 2y^2 = 100$

   (c) $8x^2 + 9y^2 = 1152$

2. $\frac{\sqrt{3}}{2}$

CHECK YOUR PROGRESS 12.2

1. $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0.$

2. $16x^2 + 9y^2 - 94x - 142y - 24xy + 324 = 0$

TERMINAL EXERCISE

1. (a) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$

   (b) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

2. (a) $\left(\pm\frac{\sqrt{5}}{6}, 0\right); \frac{\sqrt{5}}{3}$ (b) $\left(0, \pm\sqrt{21}\right); \frac{\sqrt{21}}{5}$

3. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

SENIOR SECONDARY COURSE

MATHEMATICS

Questions For Practice - Coordinate Geometry

Maximum Marks : 50 Time : $1\frac{1}{2}$ Hours

Instructions

1. Answer all questions on a separate sheet of paper.

2. Give the following information on your answer sheet:

- Name
- Enrolment Number
- Subject
- Assignment Number
- Address

3. Get your assignment checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send your assignment to the National Institute of Open Schooling

1. Show that the points (4,4), (3,5) and $(-1,1)$ are the vertices of a right angled triangle. (2)

2. Determine y when the slope of the line joining the points A $(-8,11)$ and B (2,y) is $\frac{-4}{3}$. (2)

3. Find the equation of the line passing through the points (2,1) and (7,3). (2)

4. Find the centre and diameter of the circle

$$x^2+y^2-2x+4y-11=0$$ (2)

5. Find the value of k so that the line passing through the point (k,9) and (2,7) is parallel to the line passing through the points $(2,-2)$ and (6,4). (3)

6. Find the equation of a straight line passing through the point $(3,-4)$ and perpendicular to the straight line $2x-5y+7=0$. (3)

7. Find the equation of the circle passing through the points (0,2), (2,0) and (0,0). (3)

8. Find the equation of the parabola whose focus is (2,3) and whose directrix is the line $3x+4y=1$. (3)

9. Find the locus of point which moves such that the sum of its distances from the points (3,0) and (−3,0) is less than 9. (4)

10. Find the area of a quadrilateral whose vertices, taken in order, are the points (5,−2), (4,−7), (1,1) and (3,4). (4)

11. Find the equation of the line passing through the intersection of the lines $x-3y+1=0$ and $2x+5y-9=0$ and whose distance from the origin is $\sqrt{5}$ units. (4)

12. Using coordinate geometry, prove that if two medians of a triangle are equal, then the triangle is an isosceles triangle. (6)

13. Show that the point $(a\cos\alpha, a\sin\alpha)$ lies on the circle $x^2+y^2=a^2$ for all values of α. Also show that the equation of the tangent at this point is $x\cos\alpha+y\sin\alpha=a$ and the equation of the normal is $x\sin\alpha-y\cos\alpha=0$. Further show that the line $x\sin\alpha-y\cos\alpha=a$ is also tangent to the circle for all values of α. (6)

14. Show that the line $x+y=2$ touches the circles $x^2+y^2=2$ and $x^2+y^2+3x+3y=8$ at the point where the two circles touch each other. (6)

MODULE-III

SEQUENCES AND SERIES

13. Arithmetic and Geometric Progressions
14. Special Types of Series

13

ARITHMETIC AND GEOMETRIC PROGRESSIONS

Succession of numbers of which one number is designated as the first, other as the second, another as the third and so on gives rise to what is called a sequence. Sequences have wide applications. In this lesson we shall discuss particular types of sequences called arithmetic sequence, geometric sequence and also find arithmetic mean (A.M), geometric mean (G.M) between two given numbers. We will also establish the relation between A.M and G.M.

Let us consider the following problems :

(a) A man places a pair of newly born rabbits into a warren and wants to know how many rabbits he would have over a certain period of time. A pair of rabbits will start producing offsprings two months after they were born and every following month one new pair of rabbits will appear. At the beginning the man will have in his warren only one pair of rabbits, during the second month he will have the same pair of rabbits, during the third month the number of pairs of rabbits in the warren will grow to two; during the fourth month there will be three pairs of rabbits in the warren. Thus, the number of pairs of rabbits in the consecutive months are :

1, 1, 2, 3, 5, 8, 13, ...

(b) The recurring decimal $0.\bar{3}$ can be written as a sum

$$0.\bar{3} = 0.3 + 0.03 + 0.003 + 0.0003 \ldots$$

(c) A man earns Rs.10 on the first day, Rs. 30 on the second day, Rs. 50 on the third day and so on. The day to day earning of the man may be written as 10, 30, 50, 70, 90, $\cdots$

We may ask what his earnings will be on the 10th day in a specific month.

Again let us consider the following sequences:

Notes

(1) $2, 4, 8, 16, \cdots$ (2) $\frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \ldots$

(3) $0.01, 0.0001, 0.000001, \cdots$

In these three sequences, each term except the first, progressess in a definite order but different from the order of other three problems. In this lesson we discuss those sequences whose term progressess in a definite order.

OBJECTIVES

After studying this lesson, you will be able to :

- describe the concept of a sequence (progression);
- define an A.P. and cite examples;
- find common difference and general term of a A.P;
- find the fourth quantity of an A.P. given any three of the quantities a, d, n and t_n;
- calculate the common difference or any other term of the A.P. given any two terms of the A.P;
- derive the formula for the sum of first 'n' terms of an A.P;
- calculate the fourth quantity of an A.P. given three of S, n, a and d;
- insert A.M. between two numbers;
- solve problems of daily life using concept of an A.P;
- state that a geometric progression is a sequence increasing or decreasing by a definite multiple of a non-zero number other than one;
- identify G.P.'s from a given set of progessions;
- find the common ratio and general term of a G.P;
- calculate the fourth quantity of a G.P when any three of the quantities t_n, a, r and n are given;
- calculate the common ratio and any term when two of the terms of the G.P. are given;
- write progression when the general term is given;
- derive the formula for sum of first n terms of a G.P;
- calculate the fourth quantity of a G.P. if any three of a, r, n and S are given;
- derive the formula for sum (S_∞) of infinite number of terms of a G.P. when $|r| < 1$;
- find the third quantity when any two of S_∞, a and r are given;

- convert recurring decimals to fractions using G.P;
- insert G.M. between two numbers; and
- establish relationship between A.M. and G.M.

EXPECTED BACKGROUND KNOWLEDGE

- Laws of indices
- Simultaneous equations with two unknowns.
- Quadratic Equations.

13.1 SEQUENCE

A sequence is a collection of numbers specified in a definite order by some assigned law, whereby a definite number a_n of the set can be associated with the corresponding positive integer n.

The different notations used for a sequence are.

1. $a_1, a_2, a_3, \ldots, a_n, \ldots$
2. a_n, $n = 1, 2, 3, \ldots$
3. $\{a_n\}$

Let us consider the following sequences :

1. $1, 2, 4, 8, 16, 32, \ldots$
2. $1, 4, 9, 16, 25, \ldots$
3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

In the above examples, the expression for n^{th} term of the sequences are as given below :

(1) $a_n = 2^{n-1}$ (2) $a_n = n^2$ (3) $a_n = \frac{n}{n+1}$ (4) $a_n = \frac{1}{n}$

for all positive integer n.

Also for the first problem in the introduction, the terms can be obtained from the relation

$$a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}, n \geq 3$$

A finite sequence has a finite number of terms. An infinite sequence contains an infinite number of terms.

13.2 ARITHMETIC PROGRESSION

Let us consider the following examples of sequence, of numbers :

(1) $2, 4, 6, 8, \ldots$ (2) $1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

(3) $10, 8, 6, 4, \cdots$ (4) $-\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \ldots$

Note that in the above four sequences of numbers, the first terms are respectively 2, 1, 10, and $-\frac{1}{2}$. The first term has an important role in this lesson. Also every following term of the sequence has certain relation with the first term. What is the relation of the terms with the first term in Example (1) ?

First term = 2
Second term $= 4 \quad = 2 + 1 \times 2$
Third term $= 6 \quad = 2 + 2 \times 2$
Fourth term $= 8 \quad = 2 + 3 \times 2$
and so on.

The consecutive terms in the above sequence are obtained by adding 2 to its preceding term. i.e., the difference between any two consecutive terms is the same.

A finite sequence of numbers with this property is called an arithmetic progression.

A sequence of numbers with finite terms in which the difference between two consecutive terms is the same non-zero number is called the Arithmetic Progression or simply A. P.

The difference between two consecutive terms is called the common defference of the A. P. and is denoted by 'd'.

In general, an A. P. whose first term is a and common difference is d is written as $a, a + d, a + 2d, a + 3d, \cdots$

Also we use t_n to denote the nth term of the progression.

13.2.1 GENERAL TERM OF AN A. P.

Let us consider A. P.

$$a, a + d, a + 2d, a + 3d, \cdots$$

Here, first term $(t_1) = a$

second term $(t_2) = a + d = a + (2 - 1)\, d$,

third term $(t_3) = a + 2d = a + (3 - 1)\, d$

By observing the above pattern, n^th^ term can be written as:

$$t_n = a + (n - 1)\, d$$

Hence, if the first term and the common difference of an A. P. are known then any term of A. P. can be determined by the above formula.

Sometimes, the n^{th} term of an A. P. is expressed, in terms of n, e.g. $t_n = 2n - 1$. In that case, the

A. P. will be obtained by substituting $n = 1, 2, 3, \cdots$ in the expression. In this case, the terms of the A. P. are $1, 3, 5, 7, 9, \cdots$

Note.:

(i) If the same non-zero number is added to each term of an A. P. the resulting sequence is again an A. P.

(ii) If each term of an A. P. is multiplied by the same non-zero number, the resulting sequence is again an A. P.

Example 13.1 Find the 10th term of the A. P.: 2, 4, 6, ...

Solution : Here the first term $(a) = 2$ and common difference $d = 4 - 2 = 2$

Using the formula $t_n = a + (n-1)\,d$, we have

$$t_{10} = 2 + (10 - 1)\,2 = 2 + 18 = 20$$

Hence, the 10th term of the given A. P. is 20.

Example 13.2 The 10th term of an A. P. is -15 and 31st term is -57, find the 15th term.

Solution : Let a be the first term and d be the common difference of the A. P. Then from the formula:

$t_n = a + (n-1)\,d$, we have

$$t_{10} = a + (10-1)\,d = a + 9d$$

$$t_{31} = a + (31-1)\,d = a + 30\,d$$

We have,

$$a + 9d = -15 \quad \ldots(1)$$
$$a + 30d = -57 \quad \ldots(2)$$

Solve equations (1) and (2) to get the values of a and d.

Subtracting (1) from (2), we have

$$21d = -57 + 15 = -42$$

$$\therefore \quad d = \frac{-42}{21} = -2$$

Again from (1), $a = -15 - 9d = -15 - 9(-2) = -15 + 18 = 3$

Now $t_{15} = a + (15-1)d$

$$= 3 + 14(-2) = -25$$

Example 13.3 Which term of the A. P.: 5, 11, 17, ... is 119 ?

Solution : Here $a = 5$, $d = 11 - 5 = 6$

$$t_n = 119$$

We know that

$$t_n = a + (n-1)\,d$$

$$\Rightarrow \quad 119 = 5 + (n-1) \times 6$$

$$\Rightarrow \quad (n-1) = \frac{119-5}{6} = 19$$

$$\therefore \quad n = 20$$

Therefore, 119 is the 20th term of the given A. P.

Example 13.4 Is 600 a term of the A. P.: 2, 9, 16, ...?

Solution : Here, $a = 2$, and $d = 9 - 2 = 7$.

Let 600 be the n^{th} term of the A. P. We have $t_n = 2 + (n-1)\,7$

According to the question,

$$2 + (n-1)\,7 = 600$$

$$\therefore \quad (n-1)\,7 = 598$$

$$\text{or} \quad n = \frac{598}{7} + 1$$

$$\therefore \quad n = 86\frac{3}{7}$$

Since n is a fraction, it cannot be a term of the given A. P. Hence, 600 is not a term of the given A. P.

Example 13.5 The common difference of an A. P. is 3 and the 15th term is 37. Find the first term.

Solution : Here, $d = 3$, $t_{15} = 37$, and $n = 15$

Let the first term be a. We have

$$t_n = a + (n-1)\,d$$

$$37 = a + (15-1)\,3$$

$$\text{or,} \quad 37 = a + 42$$

$$\therefore \quad a = -5$$

Thus, first term of the given A. P. is -5.

Example 13.6 If $a + b + c = 0$ and $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are. in A. P., then prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A. P.

Solution. : Since $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A. P., therefore

$$\frac{b}{c+a} - \frac{a}{b+c} = \frac{c}{a+b} - \frac{b}{c+a}$$

or, $$\left(\frac{b}{c+a}+1\right) - \left(\frac{a}{b+c}+1\right) = \left(\frac{c}{a+b}+1\right) - \left(\frac{b}{c+a}+1\right)$$

or, $$\frac{a+b+c}{c+a} - \frac{a+b+c}{b+c} = \frac{a+b+c}{a+b} - \frac{a+b+c}{c+a}$$

or, $$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a} \quad (\text{Since } a+b+c \neq 0)$$

or, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A. P.

CHECK YOUR PROGRESS 13.1

1. Find the n^{th} term of each of the following A. P's. :

   (a) 1, 3, 5, 7, $\cdots$ (b) 3, 5, 7, 9, $\cdots$

2. If $t_n = 2n + 1$, then find the A. P.

3. Which term of the A. P. $2\frac{1}{2}, 4, 5\frac{1}{2}$, ... is 31? Find also the 10th term?

4. Is -292 a term of the A. P. 7, 4, 1, -2, ...?

5. The m^{th} term of an A. P. is n and the n^{th} term is m. Show that its $(m+n)^{\text{th}}$ term is zero.

6. Three numbers are in A. P. The difference between the first and the last is 8 and the product of these two is 20. Find the numbers.

7. The n^{th} term of a sequence is $na + b$. Prove that the sequence is an A. P. with common difference a.

13.3 TO FIND THE SUM OF FIRST *n* TERMS IN AN A. P.

Let a be the first term and d be the common difference of an A. P. Let l denote the last term, i.e., the n^{th} term of the A. P. Then,

$$l = t_n = a + (n-1)d \qquad \text{... (i)}$$

Let S_n denote the sum of the first n terms of the A. P. Then

$$S_n = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l \qquad \text{... (ii)}$$

Reversing the order of terms in the R. H. S. of the above equation, we have

$$S_n = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a \qquad \text{... (iii)}$$

Adding (ii) and (ii) vertically, we get

$$2S_n = (a+l) + (a+l) + (a+l) + \ldots \text{ containing } n \text{ terms} = n(a+l)$$

i.e., $S_n = \frac{n}{2}(a+l)$

Also $S_n = \frac{n}{2}[2a + (n-1)d]$ [From (i)]

It is obvious that $t_n = S_n - S_{n-1}$

Example 13.7 Find the sum of $2 + 4 + 6 + \ldots$ n terms.

Solution.: Here $a = 2.\ d = 4 - 2 = 2$

Using the formula $S_n = \frac{n}{2}[2a + (n-1)d]$, we get

$$S_n = \frac{n}{2}[2 \times 2 + (n-1)2]$$

$$= \frac{n}{2}[2 + 2n] = \frac{2n(n+1)}{2}$$

$$= n(n+1)$$

Example 13.8 Find the sum of the sequence $2, 3, 5, 9, 8, 15, 11, \ldots$ to $(2n+1)$ terms

Solution. Let S denote the sum. Then

$$S = 2 + 3 + 5 + 9 + 8 + 15 + 11 + \ldots (2n+1) \text{ terms}$$

$$= [2 + 5 + 8 + 11 + \ldots (n+1) \text{ terms}] + [3 + 9 + 15 + 21 + \ldots n \text{ terms}]$$

$$= \frac{n+1}{2}[2 \times 2 + (n+1-1)3] + \frac{n}{2}[2 \times 3 + (n-1)6]$$

$$= \frac{n+1}{2}[4+3n] + \frac{n}{2}[6+6n-6]$$

$$= \frac{(n+1)(3n+4)}{2} + 3n^2$$

$$= \frac{1}{2}\left[3n^2 + 7n + 4 + 6n^2\right]$$

$$= \frac{1}{2}[9n^2 + 7n + 4]$$

Example 13.9 The 35th term of an A. P. is 69. Find the sum of its 69 terms.

Solution. Let a be the first term and d be the common difference of the A. P.

We have

$$t_{35} = a + (35 - 1)\, d = a + 34\, d.$$

$$\therefore \quad a + 34\, d = 69 \qquad \text{... (i)}$$

Now by the formula, $S_n = \frac{n}{2}[2a + (n-1)\, d]$

We have

$$S_{69} = \frac{69}{2}[2a + (69-1)\, d]$$

$$= 69\,(a + 34d) \qquad \text{[using (i)]}$$

$$= 69 \times 69 = 4761$$

Example 13.10 The first term of an A. P. is 10, the last term is 50. If the sum of all the terms is 480, find the common difference and the number of terms.

Solution : We have: $a = 10, l = t_n = 50, S_n = 480.$

By substituting the values of a, t_n and S_n in the formulae

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ and } t_n = a + (n-1)\, d, \text{ we get}$$

$$480 = \frac{n}{2}[20 + (n-1)\, d] \qquad \cdots \text{(i)}$$

$$50 = 10 + (n-1)\, d \qquad \cdots \text{(ii)}$$

From (ii), $(n - 1)\, d = 50 - 10 = 40$ $\qquad \cdots$ (iii)

From (i), we have

$$480 = \frac{n}{2}(20+40) \qquad \text{using (i)}$$

or, $60n = 2 \times 480$

$$\therefore \quad n = \frac{2 \times 480}{60} = 16$$

From (iii),

$$\therefore \quad d = \frac{40}{15} = \frac{8}{3} \qquad (\text{as } n-1 = 16-1 = 15)$$

Example 13.11 Let the n^{th} term and the sum of n terms of an A. P. be p and q respectively. Prove that its first term is $\left(\frac{2q-pn}{n}\right)$

Solution: In this case, $t_n = p$ and $S_n = q$

Let a be the first term of the A. P.

Now, $S_n = \frac{n}{2}(a + t_n)$

or, $\frac{n}{2}(a+p) = q$

or, $a + p = \frac{2q}{n}$

or, $a = \frac{2q}{n} - p$

$$\therefore \quad a = \frac{2q - pn}{n}$$

CHECK YOUR PROGRESS 13.2

1. Find the sum of the following A. P's.

   (a) 8, 11, 14, 17,$\cdots$ up to 15 terms (b) 8, 3, –2, –7, –12,$\cdots$ up to n terms.

2. How many terms of the A. P.: 27, 23, 19, 15, ... have a sum 95?

3. A man takes an interest-free loan of Rs. 1740 from his friend agreeing to repay in monthly

instalments. He gives Rs. 200 in the first month and diminishes his monthly instalments by Rs. 10 each month. How many months will it take to repay the loan?

4. How many terms of the progression 3, 6, 9, 12, ··

   must be taken at the least to have a sum not less than 2000?

5. In a children potato race, *n* potatoes are placed 1 metre apart in a straight line. A competitor starts from a point in the line which is 5 metre from the nearest potato. Find an expression for the total distance run in collecting the potatoes, one at a time and bringing them back one at a time to the starting point. Calculate the value *n* if the total distance run is 162 metres.

6. If the sum of first *n* terms of *a* sequence be $an^2 + bn$, prove that the sequence is an A. P. and find its common difference ?

13.4 ARITHMETIC MEAN (A. M.)

When three numbers *a*, A and *b* are in A. P., then A is called the arithmetic mean of numbers *a* and *b*. We have, $A - a = b - A$

$$\text{or,} \quad A = \frac{a+b}{2}$$

Thus, the required A. M. of two numbers *a* and *b* is $\frac{a+b}{2}$. Consider the following A. P :

3, 8, 13, 18, 23, 28, 33.

There are five terms between the first term 3 and the last term 33. These terms are called *arithmetic means* between 3 and 33. Consider another A. P. : 3, 13, 23, 33. In this case there are two arithmetic means 13, and 23 between 3 and 33.

Generally any number of arithmetic means can be inserted between any two numbers *a* and *b*. Let $A_1, A_2, A_3, ..., A_n$ be *n* arithmetic means between *a* and *b*, then.

$$a, A_1, A_2, A_3 ..., A_n, b \text{ is an A. P.}$$

Let *d* be the common difference of this A. P. Clearly it contains $(n + 2)$ terms

$$\therefore \quad b = (n+2)^{\text{th}} \text{ term}$$

$$= a + (n+1)\,d$$

$$\therefore \quad d = \frac{b-a}{n+1}$$

$$\text{Now, } A_1 = a + d \Rightarrow A_1 = \left(a + \frac{b-a}{n+1}\right) \quad ...(i)$$

Notes

$$A_2 = a+2d \Rightarrow A_2 = \left(a+\frac{2(b-a)}{n+1}\right) \qquad ...(ii)$$

$$\vdots$$

$$A_n = a+nd \Rightarrow An\left(a+\frac{n(b-a)}{n+1}\right) \qquad ...(n)$$

These are required *n* arithmetic means between *a* and *b*.

Adding (i), (ii), ..., (*n*), we get

$$A_1 + A_2 + ... + A_n = na + \dots + \frac{b-a}{n+1}[1+2+...n]$$

$$= na+\left(\frac{b-a}{n+1}\right)\left(\frac{n(n+1)}{2}\right) = na+\frac{n(b-a)}{2} = \frac{n(a+b)}{2}$$

$= n$ [Single A. M. between *a* and *b*]

Example 13.12 Insert five arithmetic means between 8 and 26.

Solution : Let A_1, A_2, A_3, A_4 and A_5 be five arithmetic means between 8 and 26.

Therefore, 8, A_1, A_2, A_3, A_4, A_5, 26 are in A. P. with $a = 8$, $b = 26$, $n = 7$

We have $26 = 8 + (7 - 1)\,d$.

$\therefore \quad d = 3$

$\therefore \quad A_1 = a + d = 8 + 3 = 11,\ A_2 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 17,\ A_4 = a + 4d = 20$

$A_5 = a + 5d = 23$

Hence, the five arithmetic means between 8 and 26 are 11, 14, 17, 20 and 23.

Example 13.13 The *'n'*, A. M's between 20 and 80 are such that the ratio of the first mean and the last mean is 1 : 3. Find the value of *n*.

Solution : Here, 80 is the $(n+2)^{th}$ term of the A. P., whose first term is 20. Let *d* be the common difference.

$\therefore \quad 80 = 20 + (n+2-1)\,d$

or, $\quad 80 - 20 = (n+1)\,d$

or, $\quad d = \dfrac{60}{n+1}$

The first A. M. $= 20+\dfrac{60}{n+1} = \dfrac{20n+20+60}{n+1}$

$$= \frac{20n+80}{n+1}$$

$$\text{The last A. M.} = 20 + n \times \frac{60}{n+1} = \frac{80n+20}{n+1}$$

We have

$$\frac{20n+80}{n+1} : \frac{80n+20}{n+1} = 1 : 3$$

or, $\dfrac{n+4}{4n+1} = \dfrac{1}{3}$

or, $4n + 1 = 3n + 12$

or, $n = 11$

∴ The number of A. M's between 20 and 80 is 11.

Example 13.14 If x, y, z are in A. P., show that

$$(x + 2y - z)\,(2y + z - x)\,(z + x - y) = 4xy\,z$$

Solution : x, y, z are in A. P.

$$\therefore \quad y = \frac{x+z}{2} \qquad \text{...(i)}$$

$\therefore \quad (x + 2y - z)\,(2y + z - x)\,(z + x - y)$

$= (x + x + z - z)\,(x + z + z - x)\,(2y - y)$ [From (i)]

$= (2x)\,(2z)\,(y) = 4x\,yz.$

= R. H. S.

CHECK YOUR PROGRESS 13.3

1. Prove that if the number of terms of an A. P. is odd then the middle term is the A. M. between the first and last terms.
2. Between 7 and 85, m number of arithmetic means are inserted so that the ratio of $(m-3)^{\text{th}}$ and m^{th} means is 11 : 24. Find the value of m.
3. Prove that the sum of n arithmetic means between two numbers is n times the single A. M. between them.
4. If the A. M. between p^{th} and q^{th} terms of an A. P., be equal and to the A. M. between r^{th} and s^{th} terms of the A. P., then show that $p + q = r + s$.

13.5 GEOMETRIC PROGRESSION

Let us consider the following sequence of numbers :

(1) $1, 2, 4, 8, 16, \cdots$ (2) $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27} \cdots$

(3) $1, -3, 9, -27, \cdots$ (4) $x, x^2, x^3, x^4, \cdots$

If we see the patterns of the terms of every sequence in the above examples each term is related to the leading term by a definite rule.

For Example (1), the first term is 1, the second term is twice the first term, the third term is 2^2 times of the leading term.

Again for Example (2), the first term is 3, the second term is $\frac{1}{3}$ times of the first term, third term is $\frac{1}{3^2}$ times of the first term.

A sequence with this property is called a *gemetric progression.*

A sequence of numbers in which the ratio of any term to the term which immediately precedes is the same non zero number (other than1), is called a geometric progression or simply G. P. This ratio is called the common ratio.

Thus, $\frac{\textit{Second term}}{\textit{First term}} = \frac{\textit{Third term}}{\textit{Second term}} =$ ..... is called the common ratio of the geometric progression.

Examples (1) to (4) are geometric progressions with the first term 1, 3, 1, x and with common ratio $2, \frac{1}{3}, -3$, and x respectively.

The most general form of a G. P. with the first term a and common ratio r is $a, ar, ar^2, ar^3, \ldots$

13.5.1 GENERAL TERM

Let us consider a geometric progression with the first term a and common ratio r. Then its terms are given by

$$a, ar, ar^2, ar^3, \ldots$$

In this case, $t_1 = a = ar^{1-1}$

$$t_2 = ar = ar^{2-1}$$

$$t_3 = ar^2 = ar^{3-1}$$

$$t_4 = ar^3 = ar^{4-1}$$

... ...

On generalisation, we get the expression for the n^{th} term as

$$t_n = ar^{n-1} \qquad \text{... (A)}$$

13.5.2 SOME PROPERTIES OF A G. P.

(i) If all the terms of a G. P. are multiplied by the same non-zero quantity, the resulting series is also in G. P. The resulting G. P. has the same common ratio as the original one.

If a, b, c, d, ... are in G. P.

then ak, bk, ck, dk ... are also in G. P. $(k \neq 0)$

(ii) If all the terms of a G. P. are raised to the same power, the resulting series is also in G. P.

Let a, b, c, d ... are in G. P.

the a^k, b^k, c^k, d^k, ... are also in G. P. $(k \neq 0)$

The common ratio of the resulting G. P. will be obtained by raising the same power to the original common ratio.

Example 13.15 Find the 6th term of the G. P.: 4, 8, 16, ...

Solution : In this case the first term $(a) = 4$

Common ratio $(r) = 8 \div 4 = 2$

Now using the formula $t_n = ar^{n-1}$, we get

$$t_6 = 4 \times 2^{6-1} = 4 \times 32 = 128$$

Hence, the 6th term of the G. P. is 128.

Example 13.16 The 4th and the 9th term of a G. P. are 8 and 256 respectively. Find the G. P.

Solution : Let a be the first term and r be the common ratio of the G. P., then

$$t_4 = ar^{4-1} = ar^3$$

$$t_9 = ar^{9-1} = ar^8$$

According to the question, $ar^8 = 256$ 10th + ... (1)

and $ar^3 = 8$ ... (2)

$$\therefore \quad \frac{ar^8}{ar^3} = \frac{256}{8}$$

or $$r^5 = 32 = 2^5$$

$$\therefore \quad r = 2$$

Again from (2), $a \times 2^3 = 8$

$$\therefore \quad a = \frac{8}{8} = 1$$

Therefore, the G. P. is

$$1, 2, 4, 8, 16, \ldots$$

Example 13.17 Which term of the G. P.: 5, –10, 20, – 40, ... is 320?

Solution : In this case, $a = 5$; $r = \frac{-10}{5} = -2$.

Suppose that 320 is the n^{th} term of the G. P.

By the formula, $t_n = ar^{n-1}$, we get

$$t_n = 5.\,(-2)^{n-1}$$

$$\therefore \quad 5.\,(-2)^{n-1} = 320 \qquad \text{(Given)}$$

$$\therefore \quad (-2)^{n-1} = 64 = (-2)^6$$

$$\therefore \quad n - 1 = 6$$

$$\therefore \quad n = 7$$

Hence, 320 is the 7th term of the G. P.

Example 13.18 If a, b, c, and d are in G. P., then show that $(a+b)^2$, $(b+c)^2$, and $(c+d)^2$ are also in G. P.

Solution. Since a, b, c, and d are in G. P.,

$$\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$$\therefore \quad b^2 = ac,\ c^2 = bd,\ ad = bc \qquad \ldots(1)$$

$$\text{Now, } (a+b)^2 (c+d)^2 = \left[(a+b)(c+d)\right]^2 = (ac + bc + ad + bd)^2$$

$$= (b^2 + c^2 + 2bc)^2 \qquad \ldots\text{[Using (1)]}$$

$$= \left[(b+c)^2\right]^2$$

$$\therefore \quad \frac{(c+d)^2}{(b+c)^2} = \frac{(b+c)^2}{(a+b)^2}$$

Thus, $(a+b)^2$, $(b+c)^2$, $(c+d)^2$ are in G. P.

CHECK YOUR PROGRESS 13.4

1. The first term and the common ratio of a G. P. are respectively 3 and $-\frac{1}{2}$. Write down the first five terms.
2. Which term of the G. P. series 1, 2, 4, 8, 16, ... is 1024? Is 520 a term of the G. P. series.?
3. Three numbers are in G. P. Their sum is 43 and their product is 216. Find the numbers in proper order.
4. The n^{th} term of a G. P. is 2×3^n for all n. Find (*a*) the first term (*b*) the common ratio of the G. P.

13.6 SUM OF *n* TERMS OF A G. P.

Let a denote the first term and r the common ratio of a G. P. Let S_n represent the sum of first n terms of the G. P.

Thus, $S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}$ ... (1)

Multiplying (1) by r, we get

$$r\,S_n = ar + ar^2 + ... + ar^{n-2} + ar^{n-1} + ar^n \qquad ...\ (2)$$

$$(1) - (2) \Rightarrow S_n - rS_n = a - ar^n$$

or $S_n(1 - r) = a(1 - r^n)$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} \qquad ....(A)$$

$$= \frac{a(r^n-1)}{r-1} \qquad ....(B)$$

Either (A) or (B) gives the sum up to the n^{th} term when $r \neq 1$. It is convenient to use formula (A) when $|r| < 1$ and (B) when $|r| > 1$.

Example 13.19 Find the sum of the G. P.:

1, 3, 9, 27, ... up to the 10^{th} term.

Solution : Here the first term $(a) = 1$ and the common ratio $(r) = \frac{3}{1} = 3$

Now using the formula, $S_n = \frac{a(r^n-1)}{r-1}$, $(\because r > 1)$ we get

$$S_{10} = \frac{1.(3^{10}-1)}{3-1} = \frac{3^{10}-1}{2}$$

Example 13.20 Find the sum of the G. P.: $\frac{1}{\sqrt{3}}, 1, \sqrt{3}, , \ldots, 81$

Solution : Here, $a = \frac{1}{\sqrt{3}}; r = \sqrt{3}$ and $t_n = l = 81$

Now $t_n = 81 = \frac{1}{\sqrt{3}}(\sqrt{3})^{n-1} = (\sqrt{3})^{n-2}$

$$\therefore \quad (\sqrt{3})^{n-2} = 3^4 = (\sqrt{3})^8$$

$$\therefore \quad n - 2 = 8$$

or $\quad n = 10$

$$\therefore \quad S_n = \frac{\frac{1}{\sqrt{3}}\left[\sqrt{3}^{10} - 1\right]}{\sqrt{3} - 1}$$

$$= \frac{\left(\sqrt{3}\right)^{10} - 1}{3 - \sqrt{3}}$$

Example 13.21 Find the sum of the G. P.: 0.6, 0.06, 0.006, 0.0006, $\cdots$ to n terms.

Solution. Here, $a = 0.6 = \frac{6}{10}$ and $r = \frac{0.06}{0.6} = \frac{1}{10}$

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we have $\quad [\because r < 1]$

$$S_n = \frac{\frac{6}{10}\left\{1 - \left(\frac{1}{10}\right)^n\right\}}{1 - \frac{1}{10}} = \frac{6}{9}\left(1 - \frac{1}{10^n}\right)$$

$$= \frac{2}{3}\left(1 - \frac{1}{10^n}\right)$$

Hence, the required sum is $\frac{2}{3}\left(1 - \frac{1}{10^n}\right)$.

Example 13.22 How many terms of the following G. P.: 64, 32, 16, $\cdots$ has the sum $127\frac{1}{2}$?

Solution : Here, $a = 64, r = \frac{32}{64} = \frac{1}{2}$ (<1) and $S_n = 127\frac{1}{2} = \frac{255}{2}$

Using the formula $S_n = \frac{a(1-r^n)}{1-r}$, we get

$$S_n = \frac{64\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{64\left\{1-\left(\frac{1}{2}\right)^n\right\}}{1-\frac{1}{2}} = \frac{255}{2} \quad \text{... (given)}$$

or $\quad 128\left[1-\left(\frac{1}{2}\right)^n\right] = \frac{255}{2}$

or $\quad 1-\left(\frac{1}{2}\right)^n = \frac{255}{256}$

or $\quad \left(\frac{1}{2}\right)^n = 1-\frac{255}{256} = \frac{1}{256} = \left(\frac{1}{2}\right)^8$

$\therefore \quad n = 8$

Thus, the required number of terms is 8.

Example 13.23 Find the sum of the following sequence :

2, 22, 222, ... to n terms.

Solution : Let S denote the sum. Then

$S = 2 + 22 + 222 + \cdots$ to n terms

$= 2\,(1 + 11 + 111 + \cdots$ to n terms)

$= \frac{2}{9}\,(9 + 99 + 999 + \cdots$ to n terms)

$$=\frac{2}{9}\left\{(10-1)+\left(10^2-1\right)+\left(10^3-1\right)+\quad ...to\ n\ terms)\right\}$$

$$=\frac{2}{9}\left\{\left(10+10^2+10^3+...to\ n\ terms\right)-\left(1+1+1+\quad ...to\ n\ terms\right)\right\}$$

$$=\frac{2}{9}\left\{\frac{(10^n-1)}{10-1}-n\right\} \qquad [\because 10+10^2+10^3+\cdots \text{ is a G P with } r=10>1]$$

$$=\frac{2}{9}\left[\frac{10^n-1-9n}{9}\right]$$

$$=\frac{2}{81}\left(10^n-1-9n\right)$$

Example 13. 24 Find the sum up to n terms of the sequence:

$$0.7,\ 0.77,\ 0.777,\ \cdots$$

Solution : Let S denote the sum, then

$$S = 0.7 + 0.77 + 0.777 + \cdots \text{ to } n \text{ terms}$$

$$= 7(0.1 + 0.11 + 0.111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\ (0.9 + 0.99 + 0.999 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{7}{9}\ \{(1-0.1) + (1-0.01) + (1-0.001) + \cdots \text{ to } n \text{ terms}\}$$

$$= \frac{7}{9}\ \{(1+1+1+ ...\ n \text{ terms}) - (0.1 + 0.01 + 0.001 + \cdots \text{ to } n \text{ terms})\}$$

$$=\frac{7}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots to\ n\ terms\right)\right\}$$

$$=\frac{7}{9}\left\{n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}\right\} \qquad (\text{Since } r<1)$$

$$=\frac{7}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$$

$$= = \frac{7}{9}\left[\frac{9n-1+10^{-n}}{9}\right]$$

$$= \frac{7}{81}\left[9n-1+10^{-n}\right]$$

CHECK YOUR PROGRESS 13.5

1. Find the sum of each of the following G. P's :

   (a) 6, 12, 24, ... to 10 terms (b) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$ ... to 20 terms.

2. How many terms of the G. P. 8, 16, 32, 64, $\cdots$ have their sum 8184 ?

3. Show that the sum of the G. P. $a + b + ... + l$ is $\frac{bl - a^2}{b - a}$

4. Find the sum of each of the following sequences up to n terms.

   (a) 8, 88, 888, ... (b) 0.2, 0.22, 0.222, ...

13.7 INFINITE GEOMETRIC PROGRESSION

So far, we have found the sum of a finite number of terms of a G. P. We will now learn to find out the sum of infinitely many terms of a G P such as.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

We will proceed as follows: Here $a = 1,\ r = \frac{1}{2}$.

The n^{th} term of the G. P. is $t_n = \frac{1}{2^{n-1}}$ and sum to n terms

i.e., $$S_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

$$= 2 - \frac{1}{2^{n-1}} < 2.$$

So, no matter, how large n may be, the sum of n terms is never more than 2.

So, if we take the sum of all the infinitely many terms, we shall not get more than 2 as answer.

Also note that the recurring decimal $0.\bar{3}$ is really $0.3 + 0.03 + 0.003 + 0.0003 + \ldots$

i.e., 0.3 is actually the sum of the above infinite sequence.

On the other hand it is at once obvious that if we sum infinitely many terms of the G. P. 1, 2, 4, 8, 16, ... we shall get a finite sum.

So, sometimes we may be able to add the infinitely many terms of G. P. and sometimes are may not. We shall discuss this question now.

13.7.1 SUM OF INFINITE TERMS OF A G. P.

Let us consider a G. P. with infinite number of terms and common ratio r.

Case 1 : We assume that $|r| > 1$

The expression for the sum of n terms of the G. P. is then given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{ar^n}{r-1} - \frac{a}{r-1} \qquad \ldots (A)$$

Now as n becomes larger and larger r^n also becomes larger and larger. Thus, when n is infinitely large and $|r| > 1$ then the sum is also infinitely large which has no importance in Mathematics. We now consider the other possibility.

Case 2 : Let $|r| < 1$

Formula (A) can be written as

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

Now as n becomes infinitely large, r^n becomes infinitely small, i.e., as $n \to \infty$, $r^n \to 0$, then the above expression for sum takes the form

$$S = \frac{a}{1 - r}$$

Hence, the sum of an infinite G. P. with the first term a and common ratio r is given by

$$S = \frac{a}{1 - r}, \text{ when } |r| < 1 \qquad \ldots \text{(i)}$$

Example 13.25 Find the sum of the infinite G. P. $\frac{1}{3}, -\frac{2}{9}, \frac{4}{27}, -\frac{8}{81}, \cdots$

Solution : Here, the first term of the infinite G. P. is $a = \frac{1}{3}$, and $r = \frac{-\frac{2}{9}}{\frac{1}{3}} = -\frac{2}{3}$.

Here, $|r| = \left|-\frac{2}{3}\right| = \frac{2}{3} < 1$

$\therefore$ Using the formula for sum $S = \frac{a}{1-r}$ we have

$$S = \frac{\frac{1}{3}}{1-\left(\frac{-2}{3}\right)} = \frac{\frac{1}{3}}{1+\frac{2}{3}} = \frac{1}{5}$$

Hence, the sum of the given G. P. is $\frac{1}{5}$.

Example 13.26 Express the recurring decimal $0.\bar{3}$ as an infinite G. P. and find its value in rational form.

Solution. $0.\bar{3} = 0.3333333 \ldots$

$$= 0.3 + 0.03 + 0.003 + 0.0003 + \ldots.$$

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ \ldots$$

The above is an infinite G. P. with the first term $a = \frac{3}{10}$ and $r = \frac{\frac{3}{10^2}}{\frac{3}{10}} = \frac{1}{10} < 1$

Hence, by using the formula $S = \frac{a}{1-r}$, we get

$$0.\bar{3} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}}$$

$$= \frac{3}{9} = \frac{1}{3}$$

Hence, the recurring decimal $0.\bar{3} = \frac{1}{3}$.

Example 13.27 The distance travelled (in cm) by a simple pendulum in consecutive seconds are 16, 12, 9, ... How much distance will it travel before coming to rest ?

Solution : The distance travelled by the pendulum in consecutive seconds are, 16, 12, 9, ... is an infinite geometric progression with the first term $a = 16$ and $r = \frac{12}{16} = \frac{3}{4} < 1$.

Hence, using the formula $S = \frac{a}{1-r}$ we have

$$S = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}} = 64$$

$\therefore$ Distance travelled by the pendulum is 64 cm.

Example 13.28 The sum of an infinite G. P. is 3 and sum of its first two terms is $\frac{8}{3}$. Find the first term.

Solution: In this problem $S = 3$. Let a be the first term and r be the common ratio of the given infinite G. P.

Then according to the question.

$$a + ar = \frac{8}{3}$$

or, $3a(1+r) = 8$ ... (1)

Also from $S = \frac{a}{1-r}$, we have

$$3 = \frac{a}{1-r}$$

or, $a = 3(1-r)$ ... (2)

From (1) and (2), we get.

$$3.3\,(1-r)\,(1+r) = 8$$

or, $1 - r^2 = \frac{8}{9}$

or, $r^2 = \frac{1}{9}$

or, $r = \pm\frac{1}{3}$

From (2), $a = 3\left(1 \mp \frac{1}{3}\right) = 2$ or 4 according as $r = \pm\frac{1}{3}$.

CHECK YOUR PROGRESS 13.6

(1) Find the sum of each of the following inifinite G. P's :

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \infty$ (b) $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots \infty$

2. Express the following recurring decimals as an infinite G. P. and then find out their values as a rational number.

(a) $0.\overline{7}$ (b) $0.3\overline{15}$

3. The sum of an infinite G. P. is 15 and the sum of the squares of the terms is 45. Find the G.P.

4. The sum of an infinite G. P. is $\frac{1}{3}$ and the first term is $\frac{1}{4}$. Find the G.P.

13.8 GEOMETRIC MEAN (G. M.)

If a, G, b are in G. P., then G is called the geometric mean between a and b.

If three numbers are in G. P., the middle one is called the geometric mean between the other two.

If $a, G_1, G_2, \ldots, G_n, b$ are in G. P.,

then $G_1, G_2, \ldots G_n$ are called n G. M.'s between a and b.

The geometric mean of n numbers is defined as the n^{th} root of their product.

Thus if $a_1, a_2, \ldots, a_n$ are n numbers, then their

$$\text{G. M.} = (a_1, a_2, \ldots a_n)^{\frac{1}{n}}$$

Let G be the G. M. between a and b, then a, G, b are in G. P.

$$\frac{G}{a} = \frac{b}{G}$$

or, $G^2 = ab$

or, $G = \sqrt{ab}$

$\therefore$ Geometric mean $= \sqrt{\text{Product of extremes}}$

Given any two positive numbers a and b, any number of geometric means can be inserted between them. Let $a_1, a_2, a_3 ..., a_n$ be n geometric means between a and b.

Then $a_1, a_1, a_3, ... a_n, b$ is a G. P.

Thus, b being the $(n + 2)^{\text{th}}$ term, we have

$$b = a\,r^{n+1}$$

or, $$r^{n+1} = \frac{b}{a}$$

or, $$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence, $$a_1 = ar = a \times \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$a_2 = ar^2 = a \times \left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\ldots \qquad \ldots$$

$$\ldots \qquad \ldots$$

$$a_n = ar^n = a \times \left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Further we can show that the product of these n G. M.'s is equal to n^{th} power of the single geometric mean between a and b.

Multiplying $a_1 . a_2, ... a_n$, we have

$$a_1, a_2 \cdots a_n = a^n \left(\frac{b}{a}\right)^{\frac{1}{n+1}+\frac{2}{n+1}\cdots+\frac{n}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{1+2+\cdots+n}{n+1}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n(n+1)}{2(n+1)}}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}} = \left(\sqrt{ab}\right)^n = G^n$$

$= (\text{single G. M. between } a \text{ and } b)^n$

Example 13.29 Find the G. M. between $\frac{3}{2}$ and $\frac{27}{2}$

Solution : We know that if G is the G. M. between a and b, then

$$G = \sqrt{ab}$$

$\therefore$ G. M. between $\frac{3}{2}$ and $\frac{27}{2} = \sqrt{\frac{3}{2} \times \frac{27}{2}} = \frac{9}{2}$

Example 13.30 Insert three geometric means between 1 and 256.

Solution : Let G_1, G_2, G_3, be the three geometric means between 1 and 256.

Then 1, G_1, G_2, G_3, 256 are in G. P.

If r be the common ratio, then $t_5 = 256$

i.e, $ar^4 = 256 \Rightarrow 1.\ r^4 = 256$

or, $r^2 = 16$

or, $r = \pm 4$

When $r = 4$, $G_1 = 1.\,4 = 4$, $G_2 = 1.\,(4)^2 = 16$ and $G_3 = 1.\,(4)^3 = 64$

When $r = -4$, $G_1 = -4$, $G_2 = (1)\,(-4)^2 = 16$ and $G_3 = (1)\,(-4)^3 = -64$

$\therefore$ G.M. between 1 and 256 are 4, 16, 64, or, – 4, 16, –64.

Example 13.31 If 4, 36, 324 are in G. P. insert two more numbers in this progression so that it again forms a G. P.

Solution : G. M. between 4 and 36 = $\sqrt{4 \times 36} = \sqrt{144} = 12$

G. M. between 36 and 324 = $\sqrt{36 \times 324} = 6 \times 18 = 108$

If we introduce 12 between 4 and 36 and 108 betwen 36 and 324, the numbers 4, 12, 36, 108, 324 form a G. P.

$\therefore$ The two new numbers inserted are 12 and 108.

Notes

Example 13.32 Find the value of n such that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Solution : If x be G. M. between a and b, then

$$x = a^{\frac{1}{2}} \times b^{\frac{1}{2}}$$

$$\therefore \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n} = a^{\frac{1}{2}}\, b^{\frac{1}{2}}$$

$$\text{or,} \quad a^{n+1}+b^{n+1} = \left(a^{\frac{1}{2}}\, b^{\frac{1}{2}}\right)\left(a^n+b^n\right)$$

$$\text{or,} \quad a^{n+1}+b^{n+1} = a^{n+\frac{1}{2}}\, b^{\frac{1}{2}} + a^{\frac{1}{2}}\, b^{n+\frac{1}{2}}$$

$$\text{or,} \quad a^{n+1} - a^{n+\frac{1}{2}}.\, b^{\frac{1}{2}} = a^{\frac{1}{2}}\, b^{n+\frac{1}{2}} - b^{n+1}$$

$$\text{or,} \quad a^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right) = b^{n+\frac{1}{2}}\left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)$$

$$\text{or,} \quad a^{n+\frac{1}{2}} = b^{n+\frac{1}{2}}$$

$$\text{or,} \quad \frac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1$$

$$\text{or,} \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}} = \left(\frac{a}{b}\right)^{0}$$

$$\therefore \quad n+\frac{1}{2} = 0 \text{ or, } n = \frac{-1}{2}.$$

13.8.1 RELATIONSHIP BETWEEN A. M. AND G.M.

Let a and b be the two numbers.

Let A and G be the A. M. and G. M. respectively between a and b

$$\therefore \quad A = \frac{a+b}{2}, G = \sqrt{ab}$$

$$A - G = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{ab}}{2}$$

$$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$$

$$\therefore \quad A > G$$

Example 13.33 The arithemetic mean between two numbers is 34 and their geometric mean is 16. Find the numbers.

Solution : Let the numbers be a and b.

Since A. M. between a and b is 34,

$$\therefore \quad \frac{a+b}{2} = 34, \text{ or, } a+b = 68 \qquad \ldots(1)$$

Since G. M. between a and b is 16,

$$\therefore \quad \sqrt{ab} = 16 \text{ or, } ab = 256$$

we know that $(a-b)^2 = (a+b)^2 - 4\,ab \qquad \cdots(2)$

$$= (68)^2 - 4 \times 256$$

$$= 4624 - 1024 = 3600$$

$$\therefore \quad a - b = \sqrt{3600} = 60 \qquad \cdots(3)$$

Adding (1) and (3), we get, $2a = 128$

$\therefore \quad a = 64$

Subtracting (3) from (1), we get

$2b = 8 \quad$ or, $\quad b = 4$

$\therefore$ Required numbers are 64 and 4.

Example 13.34 The arithmetic mean between two quantities b and c is a and the two geometric means between them are g_1 and g_2. Prove that

$$g_1^3 + g_2^3 = 2\,abc$$

Solution : The A. M. between b and c is a

$$\therefore \quad \frac{b+c}{2} = a, \text{ or, } b + c = 2a$$

Again g_1 and g_2 are two G. M.'s between b and c

$\therefore$ b, g_1, g_2, c are in G. P.

If r be the common ratio, then

$$c = br^3 \quad \text{or,} \quad r = \left(\frac{c}{b}\right)^{\frac{1}{3}}$$

$$g_1 = br = b\left(\frac{c}{b}\right)^{\frac{1}{3}}$$

and $g_2 = br^2 = b\left(\frac{c}{b}\right)^{\frac{2}{3}}$

$$\therefore \quad g_1^3 + g_2^3 = b^3\left[\left(\frac{c}{b}\right) + \left(\frac{c}{b}\right)^2\right]$$

$$= b^3 \times \frac{c}{b}\left(1 + \frac{c}{b}\right)$$

$$= b^2c \times \left(\frac{b+c}{b}\right)$$

$$= bc\,(2a) \qquad [\text{since } b + c = 2a]$$

$$= 2abc$$

Example 13.35 If one geometric mean G and two arithmetic means p and q be inserted between two quantities, show that :

$$G^2 = (2p - q)\,(2q - p)$$

Solution : Let the two quantities be a and b, then

$$G = \sqrt{ab} \text{ or, } G^2 = ab \qquad ...(i)$$

Also p and q are two A. M.'s between a and b

$\therefore$ a, p, q, b are in A. P.

$\therefore$ $p-a=q-p$ and $q-p=b-q$

$\therefore$ $a=2p-q$ and $b=2q-p$

$\therefore$ $G^2=ab=(2p-q)(2q-p)$

Example 13.36 The product of first three terms of a G. P. is 1000. If we add 6 to its second term and 7 to its 3rd term, the three terms form an A. P. Find the terms of the G. P.

Solution : Let $t_1=\frac{a}{r}, t_2=a$ and $t_3=ar$ be the first three terms of G. P.

Then, their product $=\frac{a}{r}. a. ar=1000$ or, $a^3=1000$, or, $a=10$

By the question, t_1, t_2+6, t_3+7 are in A. P. ...(1)

i.e. $\frac{a}{r}, a+6, ar+7$ are in A. P.

$$\therefore \quad (a+6)-\frac{a}{r}=(ar+7)-(a+6)$$

$$\text{or,} \quad 2(a+6)=\frac{a}{r}+(ar+7)$$

$$\text{or,} \quad 2(10+6)=\frac{10}{r}+(10r+7) \qquad \text{[using (1)]}$$

$$\text{or,} \quad 32r=10+10r^2+7r$$

$$\text{or,} \quad 10r^2-25r+10=0$$

$$\therefore \quad r=\frac{25\pm\sqrt{625-400}}{20}=\frac{25\pm 15}{20}$$

$$=2, \frac{1}{2}$$

When $a=10$, $r=2$. then the terms are $\frac{10}{2}$, 10(2) i.e., 5, 10, 20

When $a=10$, $r=\frac{1}{2}$ then the terms are 10(2), 10, $10\left(\frac{1}{2}\right)$ i.e., 20, 10, 5

CHECK YOUR PROGRESS 13.7

1. Insert 8 G. M.'s between 8 and $\frac{1}{64}$.

2. If a_1 is the first of n geometric means between a and b, show that

   $a_1^{n+1} = a^n b$

3. If G is the G. M. between a and b, prove that $\frac{1}{G^2 - a^2} + \frac{1}{G^2 - b^2} = \frac{1}{G^2}$

4. If the A. M. and G. M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$

5. If A and G are respectvely arithmetic and geometric means between two numbers a and b, then show that $A > G$.

6. The sum of first three terms of a G. P. is $\frac{13}{12}$ and their product is -1. Find the G. P.

7. The product of three terms of a G. P. is 512. If 8 added to first and 6 added to second term, the numbers form an A. P., Find the numbers.

LET US SUM UP

- The general term of an A. P.

  $a, a + d, a + 2d, \ldots$ is given by $t_n = a + (n - 1) d$

- S_n, the sum of the first n terms of an A. P.

  $a, a + d, a + 2d, \ldots$ is

  given by $S_n = \frac{n}{2}[2a + (n-1) d]$

  $= \frac{n}{2} (a + l)$, where $l = a + (n - 1) d$.

- A sequence in which the difference of two cousecutive terms is always constant $(\neq 0)$ is called an Arithmetic Progression (A. P.)

- $t_n = S_n - S_{n-1}$

- $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.
- An arithmetic mean between a and b is $\dfrac{a+b}{2}$.
- A sequence in which the ratio of two consecutive terms is always constant ($\neq 0$) is called a Geometric Progression (G. P.)
- The n^{th} term of a G. P.: $a, ar, ar^2, \ldots$ is ar^{n-1}
- Sum of the first n terms of a G. P.: $a, ar, ar^2, \ldots$ is

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ for } |r| > |$$

$$= \frac{a(1-r^n)}{1-r} \text{ for } |r| < 1$$

- The sums of an infintite G. P. $a, ar, ar^2, \ldots$ is given by

$$S = \frac{a}{1-r} \text{ for } |r| < 1$$

- Geometric mean G between two numbers a and b is $\sqrt{ab}$
- The arithmetic mean A between two numbers a and b is always greater than the corresponding Geometric mean G i.e., $A > G$.

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the sum of all the natural numbers between 100 and 200 which are divisible by 7.
2. The sum of the first n terms of two A. P.'s are in the ratio $(2n-1):(2n+1)$. Find the ratio of their 10^{th} terms.
3. If a, b, c are in A. P. then show that $b+c, c+a, a+b$ are also in A. P.

4. If $a_1, a_2, \ldots, a_n$ are in A. P., then prove that

$$\frac{1}{a_1\, a_2} + \frac{1}{a_2\, a_3} + \frac{1}{a_3\, a_4} + \ldots + \frac{1}{a_{n-1}\, a_n} = \frac{n-1}{a_1\, a_n}.$$

5. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A. P., then prove that

$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$, are also in A. P.

6. If the p^{th}, q^{th} and r^{th} terms are P, Q, R respectively. Prove that $P(Q-R) + Q(R-P) + R(P-Q) = 0$.

7. If a, b, c are in G. P. then prove that

$$a^2\, b^2\, c^2 \left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$$

8. If a, b, c, d are in G. P., show that each of the following form a G. P. :

(a) $(a^2 - b^2), (b^2 - c^2), (c^2 - d^2)$

(b) $\frac{1}{a^2 + b^2}, \frac{1}{b^2 + c^2}, \frac{1}{c^2 - d^2}$

9. If x, y, z are the p^{th}, q^{th} and r^{th} terms of a G. P., prove that

$x^{q-r}\, y^{r-p}\, z^{p-q} = 1$

10. If a, b, c are in A. P. and x, y, z are in G. P. then prove that

$x^{b-c}\, y^{c-a}\, z^{a-b} = 1$

11. If the sum of the first n terms of a G. P. is represented by S_n, then prove that

$S_n (S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$

12. If p, q, r are in A. P. then prove that the p^{th}, q^{th} and r^{th} terms of a G. P. are also in G. P.

13. If $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} \ldots + \frac{1}{2^{n-1}}$, find the least value of n such that

$$2 - S_n < \frac{1}{100}$$

14. If the sum of the first n terms of a G. P. is S and the product of these terms is p and the sum of their reciprocals is R, then prove that

$$p^2 = \left(\frac{S}{R}\right)^n$$

ANSWERS

CHECK YOUR PROGRESS 13.1

1. (a) $2n-1$ (b) $2n+1$

2. 3, 5, 7, 9, ... 3. 20, 16 4. no 5. $m+n$ 6. 10, 6, 2, ...

CHECK YOUR PROGRESS 13.2

1. (a) 435 (b) $\frac{n}{2}\left[21-5n^2\right]$

2. 5 3. 12

4. 37 5. $n^2+9n, 9$

6. $2a$

CHECK YOUR PROGRESS 13. 3

2. 5

CHECK YOUR PROGRESS 13. 4

1. $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}$ 2. 11th, no

3. 36, 6, 1 or 1, 6, 36 4. (a) 6 (b) 3

CHECK YOUR PROGRESS 13.5

1. (a) 6138 (b) $\frac{2}{3}\left(1-\frac{1}{2^{20}}\right)$

2. 10.

4. (a) $\frac{80}{81}\left(10^n-1\right)-\frac{8n}{9}$ (b) $\frac{2n}{9}-\frac{2}{81}\left(1-\frac{1}{10^n}\right)$

CHECK YOUR PROGRESS 13. 6

1. (a) $\frac{3}{2}$ (b) $\frac{13}{24}$

2. (a) $\frac{7}{9}$ (b) $\frac{52}{165}$

3. $5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \ldots \infty$

4. $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \ldots \infty$

CHECK YOUR PROGRESS 13.7

1. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32},$

6. $\frac{4}{3}, -1, \frac{3}{4}$ or $\frac{3}{4}, -1, \frac{4}{3}$

7. 4, 8, 16

TERMINAL EXERCISE

1. 2107

2. 37 : 39

14

SPECIAL TYPES OF SERIES

Suppose you are asked to collect pebbles every day in such a way that on the first day if you collect one pebble, second day you collect double of the pebbles that you have collected on the first day, third day you collect double of the pebbles that you have collected on the second day, and so on. Then you write the number of pebbles collected daywise, you will have a sequence,

$1, 2, 2^2, 2^3, ...$

From a sequence we derive a series. The series corresponding to the above sequence is

$1 + 2 + 2^2 + 2^3 + ...$

One well known series is Fibonacci series

$1 + 1 + 2 + 3 + 5 + 8 + 13 + ...$

In this lesson we shall study some special types of series in detail.

OBJECTIVES

After studying this lesson, you will be able to :

- define a series;
- calculate the terms of a series for given values of n from t_n;
- evaluate $\sum n, \sum n^2, \sum n^3$ using method of differences and mathematical induction; and
- evaluate simple series like $1.3 + 3.5 + 5.7 + ...$ n terms.

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a sequence
- Concept of A. P. and G. P., sum of n terms.
- Knowldge of converting recurring decimals to fractions by using G. P.

14.1 SERIES

An expression of the form $u_1 + u_2 + u_3 + ... + u_n + ...$ is called a series, where $u_1, u_2, u_3 ..., u_n$ ... is a sequence of numbers. The above series is denoted by $\sum_{r=1}^{n} u_r$. If n is finite then the series is a finite series, otherwise the series is infinite. Thus we find that a series is associated to a sequence. Thus a series is a sum of terms arranged in order, according to some definite law.

Consider the following sets of numbers :

(a) $1, 6, 11, ...$ (b) $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, ...$

(c) $48, 24, 12, ...$ (d) $1^2, 2^2, 3^2, ...$

(a), (b), (c), (d) form sequences, since they are connected by a definite law. The series associated with them are :

$$1 + 6 + 11 + ...$$

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + ...$$

$$48 + 24 + 12 + ...$$

$$1^2 + 2^2 + 3^2 + ...$$

Example 14.1 Write the first 6 terms of each of the following sequences, whose n^th^ term is given by

(a) $T_n = 2n + 1$, (b) $a_n = n^2 - n + 1$ (c) $f_n = (-1)^n . 5^n$

Hence find the series associated to each of the above sequences.

Solution :

(a) $T_n = 2n + 1$

For $n = 1$, $T_1 = 2.1 + 1 = 3$

For $n = 2$, $T_2 = 2.2 + 1 = 5$

For $n = 3$, $T_3 = 2.3 + 1 = 7$

For $n = 4$, $T_4 = 2.4 + 1 = 9$

For $n = 5$, $T_5 = 2.5 + 1 = 11$

For $n = 6$, $T_6 = 2.6 + 1 = 13$

Hence the series associated to the above sequence is $3 + 5 + 7 + 9 + 11 + 13 + ...$

(b) $a_n = n^2 - n + 1$

For $n = 1$, $a_1 = 1^2 - 1 + 1 = 1$

For $n = 2$, $a_2 = 2^2 - 2 + 1 = 3$

For $n = 3$, $a_3 = 3^2 - 3 + 1 = 7$

For $n = 4$, $a_4 = 4^2 - 4 + 1 = 13$

For $n = 5$, $a_5 = 5^2 - 5 + 1 = 21$

For $n = 6$, $a_6 = 6^2 - 6 + 1 = 31$

Hence the series associated to the above sequence is $1 + 3 + 7 + 13 + ...$

(c) Here $f_n = (-1)^n\, 5^n$

For $n = 1$, $f_1 = (-1)^1\, 5^1 = -5$

For $n = 2$, $f_2 = (-1)^2\, 5^2 = 25$

For $n = 3$, $f_3 = (-1)^3\, 5^3 = -125$

For $n = 4$, $f_4 = (-1)^4\, 5^4 = 625$

For $n = 5$, $f_5 = (-1)^5\, 5^5 = -3125$

For $n = 6$, $f_6 = (-1)^6\, 5^6 = 15625$

The corresponding series relative to the sequence

$f_n = (-1)^n\, 5^n$ is $-5 + 25 - 125 + 625 - 3125 + 15625 -$

Example 14.2 Write the n^{th} term of each of the following series :

(a) $-2 + 4 - 6 + 8 - ....$ (b) $1 - 1 + 1 - 1 + ...$

(c) $4 + 16 + 64 + 256 + ...$ (d) $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + ...$

Solution :

(a) The series is $-2 + 4 - 6 + 8$ ....

Here the odd terms are negative and the even terms are positive. The above series is obtained by multiplying the series.

$-1 + 2 - 3 + 4 - ....$ by 2

$\therefore$ $T_n = 2(-1)^n n = (-1)^n 2n$

(b) The series is $1 - 1 + 1 - 1 + 1 -$ ......

$\therefore$ $T_n = (-1)^{n+1}$

(c) The series is $4 + 16 + 64 + 256 + \ldots$

The above series can be writen as

$$4 + 4^2 + 4^3 + 4^4 + \ldots$$

i.e., n^{th} term, $T_n = 4^n$.

(d) The series is $\sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \ldots$

i.e., $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots$

$\therefore$ n^{th} term is $T_n = \sqrt{n+1}$.

CHECK YOUR PROGRESS 14.1

1. Write the first 6 terms of each of the following series, whose n^{th} term is given by

(a) $T_n = \dfrac{n(n+1)(n+2)}{6}$ (b) $a_n = \dfrac{n^2-1}{2n-3}$

2. If $A_1 = 1$ and $A_2 = 2$, find A_6 if $A_n = \dfrac{A_{n-1}}{A_{n-2}}, (n > 2)$

3. Write the n^{th} term of each of the following series:

(a) $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ (b) $3 - 6 + 9 - 12 + \ldots$

14.2 SUM OF THE POWERS OF THE FIRST *n* NATURAL NUMBERS

(a) The series of first *n* natural numbers is

$$1 + 2 + 3 + 4 + \ldots + n.$$

Let $S_n = 1 + 2 + 3 + \ldots + n$

This is an arithmetic series whose the first term is 1, the common difference is 1 and the number of terms is *n*.

$$\therefore \quad S_n = \frac{n}{2}[2.1 + (n-1)1] = \frac{n}{2}[2 + n - 1]$$

i.e., $S_n = \dfrac{n(n+1)}{2}$

$\therefore$ We can write $\sum n = \dfrac{n(n+1)}{2}$

(b) Determine the sum of the squares of the first *n* natural numbers.

Let $S_n = 1^2 + 2^2 + 3^2 + ... + n^2$

Consider the identity :

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

By giving the values for $n = 1, 2, 3, ..., n-1, n$ in the above identity, we have.

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$

$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$

$$3^3 - 2^3 = 3.3^2 - 3.3 + 1$$

....

....

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

Adding these we get

$$n^3 - 0^3 = 3\,(1^2 + 2^2 + 3^2 + ... + n^2) - 3\,(1 + 2 + 3 + ... + n) +$$

$$(1 + 1 + 1 + ... \; n \text{ times})$$

or, $n^3 = 3\,S_n - 3\left[\dfrac{n(n+1)}{2}\right] + n .... \left[\because \sum n = \dfrac{n(n+1)}{2}\right]$

or, $3\,S_n = n^3 + \dfrac{3n(n+1)}{2} - n$

$$= n(n^2 - 1) + \frac{3n}{2}(n+1)$$

$$= n(n+1)\left(n - 1 + \frac{3}{2}\right)$$

$$= \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6}$$

i.e., $\sum n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(c) Determine the sum of the cubes of the first *n* natural numbers.

Here $S_n = 1^3 + 2^3 + 3^3 + ... + n^3$

Consider the identity : $n^4-(n-1)^4=4n^3-6n^2+4n-1$

Putting successively 1, 2, 3, ... for n we have

$$1^4-0^4=4.1^3-6.1^2+4.1-1$$

$$2^4-1^4=4.2^3-6.2^2+4.2-1$$

$$3^4-2^4=4.3^3-6.3^2+4.3-1$$

$$\ldots \qquad \ldots$$

$$n^4-(n-1)^4=4.n^3-6.n^2+4.n-1$$

Adding these, we get

$$n^4-0^4= 4(1^3+2^3+\ldots+n^3)-6(1^2+2^2+\ldots+n^2)+4\,(1+2+3+\ldots+n) - (1+1+\ldots\ n\text{ times})$$

$$\Rightarrow \quad n^4=4.S_n-6\left[\frac{n\,(n+1)\,(2n+1)}{6}\right]+4n\,\frac{n+1}{2}-n$$

$$\Rightarrow \quad 4S_n = n^4+n(n+1)\,(2n+1)-2n\,(n+1)+n$$

$$= n^4+n\,(2n^2+3n+1)-2n^2-2n+n$$

$$= n^4+2n^3+3n^2+n-2n^2-2n+n$$

$$= n^4+2n^3+n^2$$

$$= n^2\,(n^2+2n+1)$$

i.e., $4S_n = n^2\,(n+1)^2$

$$\therefore \quad S_n=\frac{n^2\,(n+1)^2}{4}=\left\{\frac{n\,(n+1)}{2}\right\}^2$$

$$\therefore \quad \sum n^3=\left[\frac{n(n+1)}{2}\right]^2$$

or, $\sum n^3=(\sum n)^2$

> *Note : In problems on finding sum of the series, we shall find the nth term of the series (t_n) and then use $S_n=\sum t_n$*

Example 14.3 Find the sum of first n terms of the series $1.3+3.5+5.7+\ldots$

Solution :

Let $S_n=1.3+3.5+5.7+\ldots$

The n^{th} term of the series

$$t_n = \{n^{th} \text{ term of } 1, 3, 5, ...\} \times \{n^{th} \text{ term of } 3, 5, 7,...\}$$

$$= (2n - 1)(2n + 1)$$

$$= 4n^2 - 1$$

$$S_n = \sum t_n$$

$$= \sum \left[4n^2 - 1\right]$$

$$= 4\sum n^2 - \sum (1)$$

$$= 4\,\frac{n(n+1)(2n+1)}{6} - n$$

$$= \frac{2n(n+1)(2n+1) - 3n}{3}$$

$$= \frac{n}{3}\left[2(2n^2 + 3n + 1) - 3\right]$$

$$= \frac{n}{3}\left[4n^2 + 6n - 1\right]$$

Example 14.4 Find the sum of first n terms of the series

$$1.2^2 + 2.3^2 + 3.4^2 + \ldots\ldots$$

Solution : Here $t_n = n\,\{2 + (n - 1)\}^2$

$$= n(n + 1)^2 = n.(n^2 + 2n + 1)$$

i.e., $\quad t_n = n^3 + 2n^2 + n$

Let $\quad S_n = 1.2^2 + 2.3^2 + 2.3^2 + 3.4^2 + ... + n.(n + 1)^2.$

$\therefore \quad S_n = \Sigma t_n = \Sigma (n^3 + 2n^2 + n)$

$$= \Sigma n^3 + 2\,\Sigma n^2 + \Sigma n.$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{n(n+1)}{2}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{3} + \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{12}(3n^2 + 11n + 10)$$

$$= \frac{1}{12} n(n+1)(n+2)(3n+5)$$

Example 14.5 Find the sum of first n terms of the series

$$2.\,3.\,5 + 3.\,5.\,7 + 4.\,7.\,9 + ...$$

Solution : Let $S_n = 2.\,3.\,5. + 3.\,5.\,7 + 4.\,7.\,9 + ...$

n^{th} term of the series

$t_n =$ {n^{th} term of 2, 3, 4, ...} × {n^{th} term of 3, 5, 7, ...} × {n^{th} term of 5, 7, 9, ...}

$$= (n+1) \times (2n+1) \times (2n+3)$$

$$= (n+1)[4n^2 + 8n + 3]$$

$$= 4n^3 + 12n^2 + 11n + 3$$

$$\therefore \quad S_n = \Sigma t_n = \Sigma [4n^3 + 12n^2 + 11n + 3]$$

$$= 4\,\Sigma n^3 + 12\,\Sigma n^2 + 11\,\Sigma n + \Sigma(3)$$

$$= 4\frac{n^2(n+1)^2}{4} + \frac{12\,n(n+1)(2n+1)}{6} + \frac{11\,n(n+1)}{2} + 3n$$

$$= n^2(n+1)^2 + 2n(n+1)(2n+1) + \frac{11n(n+1)}{2} + 3n$$

$$= \frac{n}{2}\left[2n(n+1)^2 + 4(n+1)(2n+1) + 11(n+1) + 6\right]$$

$$= \frac{n}{2}\left[2n(n^2 + 2n + 1) + 4(2n^2 + 3n + 1) + 11n + 17\right]$$

$$= \frac{n}{2}\left[2n^3 + 12n^2 + 25n + 21\right]$$

Example 14.6 Find the sum of first n terms of the following series :

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$$

Solution : $t_n = \dfrac{1}{(2n-1)(2n+1)}$

$$=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Now putting successively for $n = 1, 2, 3, \ldots.$

$$t_1=\frac{1}{2}\left[1-\frac{1}{3}\right]$$

$$t_2=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{5}\right]$$

$$t_3=\frac{1}{2}\left[\frac{1}{5}-\frac{1}{7}\right]$$

$$\cdots \qquad \cdots$$

$$t_n=\frac{1}{2}\left[\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\right]$$

Adding, $$t_1+t_2+\cdots+t_n=\frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

$$=\frac{n}{(2n+1)}$$

CHECK YOUR PROGRESS 14.2

1. Find the sum of first n terms of each of the following series :

   (a) $1+(1+3)+(1+3+5)+\ldots$

   (b) $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$

   (c) $(1)+(1+3)+(1+3+3^2)+(1+3+3^2+3^3)+\ldots$

2. Find the sum of n terms of the series. whose n^{th} term is $n(n+1)(n+4)$

3. Find the sum of the series $1.2.3+2.3.4.+3.4.5+\cdots$ upto n terms

LET US SUM UP

- An expression of the form $u_1+u_2+u_3+\ldots..+u_n+\ldots$ is called a series, where $u_1, u_2, u_3 \ldots. u_n, \ldots.$ is a sequences of numbers

- $$\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$$

- $$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{r=1}^{n} r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$
- $$S_n = \Sigma t_n$$

SUPPORTIVE WEB SITES

http://www.wikipedia.org

http://mathworld.wolfram.com

TERMINAL EXERCISE

1. Find the sum of each of the following series :

   (a) $2 + 4 + 6 + \dots$ up to 40 terms. (b) $2 + 6 + 18 + \dots$ up to 6 terms.

2. Sum each of the following series to n terms :

   (a) $1 + 3 + 7 + 15 + 31 + \dots\dots$ (b) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \dots\dots$

   (c) $\frac{3}{1.4} + \frac{5}{4.9} + \frac{7}{9.16} + \frac{9}{16.25} + \dots$

3. Find the sum of first n terms of the series $1^2 + 3^2 + 5^2 + \dots$

4. Find the sum to n terms of the series $5 + 7 + 13 + 31 + \dots$

5. Find the sum to n terms of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$

6. Find the sum of $2^2 + 4^2 + 6^2 + \dots + (2n)^2$

7. Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \dots\dots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \dots\dots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

ANSWERS

CHECK YOUR PROGRESS 14.1

1. (a) $1, 4, 10, 20, 35, 56$

   (b) $0, 3, \frac{8}{3}, 3, \frac{24}{7}, \frac{35}{9}$

2. $\frac{1}{2}$

3. (a) $(-1)^n \frac{1}{n}$

   (b) $(-1)^{n+1} 3n$

CHECK YOUR PROGRESS 14.2

1. (a) $\frac{1}{6} n(n+1)(2n+1)$

   (b) $\frac{n}{3n+1}$

   (c) $\frac{1}{4}(3^{n+1} - 2n - 3)$

2. $\frac{n(n+1)}{12}\left[3n^2 + 23n + 34\right]$

3. $\frac{1}{4} n(n+1)(n+2)(n+3)$

TERMINAL EXERCISE

1. (a) 1640

   (b) 728

2. (a) $2^{n+1} - n - 2$

   (b) $\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$

(c) $1-\frac{1}{(n+1)^2}$

3. $\frac{n}{3}(4n^2-1)$

4. $\frac{1}{2}(3^n+8n-1)$

5. $\frac{5}{4}+\frac{15}{16}\left(1-\frac{1}{5^{n-1}}\right)-\frac{3n-2}{4.(5^{n-1})}$

6. $\frac{2n(n+1)(2n+1)}{3}$

SENIOR SECONDARY COURSE

MATHEMATICS

Questions For Practice - Sequences and Series

Maximum Marks : 50 Time : $1\frac{1}{2}$ Hours

Instructions

1. Answer all questions on a separate sheet of paper.
2. Give the following information on your answer sheet:
   - Name
   - Enrolment Number
   - Subject
   - Assignment Number
   - Address
3. Get your assignment checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send your assignment to the National Institute of Open Schooling

1. Find 10th term of the A.P.

   $1, 3, 5, \cdots$ (2)

2. Which term of the A.P.

   $2, 5, 8, 11, \cdots$ is 44? (2)

3. Fifth term of a G.P. is $\frac{1}{3}$ and 9th term is $\frac{16}{243}$. Find the common ratio. (2)

4. Find 4th term from the end of the G.P.

   $3, 6, 12, 24, \cdots, 3072$ (2)

5. How many terms of the sequence

   $-2, 3, 8, 13, \cdots$

   will make the sum 568? (3)

6. Find the sum of 12 terms of the series

$$\sqrt{2}+2+2\sqrt{2}+4+\cdots$$

(3)

7. If 4th terms of an A.P. is equal to 3 times the first term, and 7th term exceeds twice its third term by 1, find the first term and common difference.

(3)

8. If pth, qth, rth terms of a G.P. be $x,y,z,$ respectively, prove that

$$x^{q-r}.y^{r-p}.z^{p-q}=1$$

(3)

9. Find the sum of all natural numbers between 1and 99 which are multiples of 5. (4)

10. Find the sum to n terms of the series 0.3 + 0.33 + 0.333 + … (4)

11. Find the sum to infinity of the series

$$1+3r+9r^2+27r^3+\cdots,|r|<\frac{1}{3}$$

(4)

12. If a, b, c, are in A.P. show that

$$\frac{a(b+c)}{bc},\quad\frac{b(c+a)}{ca},\quad\frac{c(a+b)}{ab}$$

are also in A.P.

(6)

13. If S_1, S_2, S_3 be the sum of n, $2n$ and $3n$ terms of a G.P. respectively, prove that

$$S_1(S_3-S_2)=(S_1-S_2)^2$$

(6)

14. If the sum to infinity of the series

$$3+5r+7r^2+\cdots$$

is $\frac{44}{9}$, find r.

(6)

CURRICULUM IN MATHEMATICS AT SENIOR SECONDARY LEVEL

RATIONALE

The Senior Secondary stage is most crucial where learners for the first time move towards diversification. At this stage, the learners start thinking to take important decisions concerning their future career by choosing suitable courses. It is the stage, from where learners would either go for higher academic education in Mathematics or for Professional courses or it may be the end of their academic career. In view of these facts, it is important to make Mathematics Education at this level broad based and meaningful. The revised curriculum in Mathematics has been designed to meet the needs of all categories of learners. In order to relate Mathematics to real life and work situations of NIOS learners, greater emphasis has been put on applications of various concepts.

COURSE OBJECTIVES

The basic aims of revising the course material at Senior Secondary level are to:

- acquire knowledge and understanding of basic concepts, facts, principles, terms, symbols in the learning material;
- apply mathematical knowledge and skills to solve problems;
- introduce learners to different ways of processing given data and help them in arriving at conclusions;
- provide learners with an appreciation of the wide variety of application of Mathematics, and equip them with the basic tools that enable such application;
- develop appreciation for the power and beauty of Mathematics for its applications in Science, Commerce, Economics and daily life;
- develop positive attitude towards Mathematics and its application.
- include new and emerging areas in Mathematics along with contribution of Mathematicians thereto;

COURSE STRUCTURE

The present curriculum in Mathematics has been divided into six core modules namely, **Algebra, Coordinate Geometry, Sequences and Series, Functions ,Calculus** and **Statistics.** Besides, the learner have to choose any one of the two optional modules namely, **Vectors and Three Dimensional Geometry,** and **Mathematics for Commerce, Economics and Business.** Each module has been divided further into units and then into lessons.

The number of lessons, suggested study time and marks allotted for each unit is as follows:

Unitwise Distribution of Core Modules	Number of Lessons	Minimum Study Time (in Hours)	Marks	
			to each Unit	to each Module
MODULE 1 : ALGEBRA				22
1.1 Complex Numbers and Quadratic Equations	2	12	6	
1.2 Matrices and Determinants	3	16	6	
1.3 Inequations and their Applications	1	8	4	
1.4 Permutations, Combinations and Binomial Theorem	2	12	6	
MODULE 2 : COORDINATE GEOMETRY				10
2.1 System of Coordinates and Straight Lines	2	16	6	
2.2 Circles and Conic Sections	2	12	4	
MODULE 3 : SEQUENCES AND SERIES				8
3.1 Sequences	1	10	4	
3.2 Series	1	10	4	
MODULE 4 : FUNCTIONS				12
4.1 Sets, Relations and Functions	1	12	4	
4.2 Trigonometric Functions	4	20	8	
MODULE 5 : CALCULUS				26
5.1 Differential Calculus	6	38	14	
5.2 Integral Calculus	3	30	12	
MODULE 6 : STATISTICS				7
6.1 Measures of Dispersion	1	6	3	
6.2 Random Experiments and Probability	2	12	4	
Unitwise Distribution of Optional Modules (You have to choose any one of the following two modules)				
MODULE 7: VECTORS AND THREE DIMENSIONAL GEOMETRY				
7.1 Vectors	1	6	5	
7.2 Three Dimensional Geometry	4	20	10	
OR				15 each
MODULE 8: MATHEMATICS FOR COMMERCE, ECONOMICS AND BUSINESS				
8.1 Mathematics for Business	4	20	10	
8.2 Application of Calculus in Commerce and Economics	1	6	5	
TOTAL	**36**	**240**		**100**

COURSE DESCRIPTION

MODULE -1: ALGEBRA

Approach: The system of real numbers is not sufficient to solve all algebraic equations. There are no real numbers which satisfy the equation $x^2+1=0$ i.e., $x^2=-1$. In order to solve such equations i.e., to find the square roots of negative numbers, we extend the system of numbers and introduce a new system of numbers known as complex numbers. Solution of a quadratic equation with real coefficients is discussed leading to relation between roots and coefficients. Cube roots of unity are also discussed.

To store some information, we use a rectangular array of numbers called a matrix. In the unit on ***"Matrices and Determinants"***, we study the algebra of matrices and distinguish it from the algebra of numbers. The study is used to solve a system of linear equations, especially when the number of unknowns is large. We study that to each square matrix a determinant is associated as a number and also study various properties of determinant. The study is used to solve a system of linear equations using Cramer's rule.

In the unit on ***"Linear Inequations and their Applications"***, the learners are introduced to the concept of linear programming and the solution of a problem through graphical representation.

In the unit on ***"Permutations, Combinations and Binomial Theorem"***, Principle of Mathematical Induction is stated and used to prove binomial theorem for a positive integral index. The concepts of permutations and combinations are used to solve daily life problems.

Unit 1.1 Complex Numbers and Quadratic Equations

Study Time: 12 Hours. Max. Marks: 6

Pre-requisites: Real numbers and quadratic equations with real coefficients.

1.1.1 Complex Numbers

- Definition in the form x + iy
- Real and imaginary parts of a complex number
- Equality of complex numbers
- Representation of a complex number by a point in a plane
- Representation of a complex number in polar form
- Modulus and argument of a complex number
- Conjugate of a complex number
- Algebraic operations on complex numbers(addition, subtraction, multiplication and division)
- Properties of algebraic operations (closure, commutativity, associativity, identity, inverse, distributivity)
- Elementary properties of modulus

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- Algebra of Complex Numbers
- Modulus and Argand Diagram

1.1.2 Quadratic Equations

- Solution of a quadratic equation with real coefficients by factorization and by using the quadratic formula
- Relation between roots and coefficients
- Cube roots of unity

Unit 1.2 Matrices and Determinants

Study Time: 16 Hours. Max. Marks: 6

Pre-requisites: Knowledge of number system; solution of system of linear equations.

1.2.1 Matrices

- Matrix as a rectangular array of numbers
- Order of a matrix (upto 3 x 3)
- Equality of two matrices
- Square and rectangular matrices
- Unit matrix, zero matrix, diagonal, row and column matrices
- Transpose of a matrix
- Symmetric and skew symmetric matrices
- Sum and difference of matrices

- Multiplication of a matrix by a scalar
- Multiplication of matrices

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- Matrices and operations on them

1.2.2 Determinants and their Applications

- Determinant of a square matrix
- Minors and Cofactors
- Evaluation of a determinant
- Properties of determinants
- Conditions for solvability of a system of linear equations
- Solution of a system of linear equations by Cramer's Rule

1.2.3 Inverse of a Matrix and its Application

- Minors and Cofactors of an element of a square matrix
- Adjoint of a matrix
- Inverse of a matrix
- Solution of a system of linear equations by matrix method

Unit 1.3 Inequations and their Applications

Study Time: 8 Hours Max. Marks: 4

Pre-requisites: Systems of Linear Equations, Coordinate System and Inequalities

1.3.1 Linear Inequations and their Applications

- Represention of linear inequations in two variable(s) graphically in a plane
- Graphical solution of a system of linear inequations in two variables
- Introduction to related terminology such as objective function, constraints, feasible region, optimization
- Formulation of different types of problems related to Diet, Business, Transportation and their graphical solutions

Supportive video Programme

- Linear Inequations and their Applications

Unit 1.4 Permutations, Combinations and Binomial Theorem

Study Time: 12 Hours Max. Marks: 6

Pre-requisites: Number System

1.4.1 Permutations and Combinations

- Fundamental Principle of Counting
- Meaning and expression of ${}^nP_r$
- Application of ${}^nP_r$
- Meaning and expression of ${}^nC_r$
- Properties of ${}^nC_r$ namely,

(i) $${}^nC_r = \frac{{}^nP_r}{n!}$$

(ii) $${}^nC_r = {}^nC_{n-r}$$

(iii) $${}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$$

- Application of ${}^nC_r$

1.4.2 Binomial Theorem

- Principle of mathematical induction and its applications
- Binomial theorem for a positive integral index (with proof) and its application.
- Binomial theorem for rational index (without proof) and its application.

MODULE -2: COORDINATE GEOMETRY

Approach: The general equation of the first degree in two variables x and y represents a straight line and is expressed in different forms. Some important results of Euclidean Geometry are proved with the help of Coordinates. The study of coordinate geometry is extended to studying ***"Conic Sections"*** as plane sections of a right circular cone, in the form of an ellipse, a circle, a parabola, a hyperbola and pair of straight lines, by taking different positions of the plane. This module has been designed to discuss standard equations of circle, ellipse and parabola in detail.

Unit 2.1 System of Coordinates and Straight Lines

Study Time: 16 Hours Max. Marks: 6

Pre-requisites: Number system and plotting of points in a plane.

2.1.1 Cartesian System of Coordinates

- Recall of Cartesian system of coordinates in a plane
- Distance Formula
- Section Formula
- Area of a Triangle
- Collinearity of three points
- Slope of a line
- Parallel and perpendicular lines
- Intercepts of a line on axes
- Locus and its equation

Supportive video Programme

- Coordinate Geometry (Part I)

2.1.2 Straight Lines

- Equation of a straight line parallel to either of the axes
- Equation(s) of a straight line in :
  - (i) slope-intercept form
  - (ii) two point form
  - (iii) point-slope form
  - (iv) intercept form
  - (v) parametric form
  - (vi) perpendicular form
- General equation of first degree in two variables i.e. $Ax + By + C = 0$ and expressing it in various forms
- Intersection of lines
- Angle between two lines
- Parallel lines
- Perpendicular lines
- Conditions for parallelism and perpendicularity
- Condition of concurrency of three lines
- Distance of a point from a line
- Distance between two parallel lines
- Application of coordinate geometry to prove some geometrical results.

Supportive video Programme

- Coordinate Geometry (Part II)

Unit 2.2 Circles and Conic Sections

Study Time: 12 Hrs Max. Marks :4

Pre-requisites: Coordinate system and straight lines.

2.2.1 Circles

- Equation of a circle with given radius and centre
- General equation of a circle
- Equation of a circle in terms of extremities of a diameter
- Parametric representation of a circle

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- Circles

2.2.2 Conic Sections

- General conic section
- Equation of parabola and ellipse in standard form
- Eccentricity, directrix and focus of a given conic section.

Supportive video Programme

- Conic Sections

MODULE -3: SEQUENCES AND SERIES

Approach: The learners are encouraged to recognize different number patterns and then introduced to the concept of a sequence as a collection of numbers obtained by certain rule. The sequence will be categorized based on the rule applied. A series is a summation of various terms of a sequence. A series could be finite or infinite, depending on the number of terms. The concept of G.P. is used to convert recurring decimal to fraction. This module has been designed to discuss some important series namely $\sum n, \sum n^2, \sum n^3$ etc. using method of differences and mathematical induction.

Unit 3.1 Sequences

Study Time: 10 Hours Max. Marks: 4

Pre-requisites: Number system, Permutations and Combinations

3.1.1 Arithmetic and Geometric Progressions

- Concept of a sequence
- A.P as a sequence
- General term of an A.P
- Sum upto 'n' terms of an A.P.
- Arithmetic Mean (A.M)
- Insertion of A.M between two given numbers
- G.P as a sequence
- General term of a G.P
- Sum upto 'n' terms of a G.P
- Sum to infinite number of terms of a G.P
- Geometric Mean (G.M)
- Insertion of G.M between two given numbers
- Relation between A.M. and G.M
- Conversion of recurring decimals to fractions using G.P.

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- Progressions

Unit 3.2 Series

Study Time: 10 Hours Max. Marks: 4

Pre-requisites : Progressions

3.2.1 Special Types of Series

- Concept of a series
- Arithmetico- Geometric Series
- Some important series: $\sum n, \sum n^2, \sum n^3$ etc. using method of differences and mathematical induction

MODULE -4 : FUNCTIONS

Approach: Every branch of science and engineering offers abundant examples of variable quantities that are interrelated so that the value of one quantity determines the value of another variable. The area of a circle is completely determined by the value of the radius : $A = \pi r^2$ (where A stands for area). Temperatures are often measured in degrees Celsius or degrees Fahrenheit. On the Fahrenheit scale, water freezes at $32^\circ F$ and boils at $212^\circ F$. The Celsius scale is such that water freezes at $0^\circ C$ and boils at $100^\circ C$. The relationship between Celsius and Fahrenheit scale, given by $C = \frac{5}{9}(F - 32)$ converts a temperature of degrees Fahrenheit to degrees Celsius. All these examples, despite the diversity of meanings of the quantities involved, have one thing in common: specifying the value of one of the two variables determines the value of the other variable. A relationship of this kind between two variables is called a functional relationship. The module has been designed to highlight this concept in detail.

Some important functions like algebraic, trigonometric, exponential and logarithmic functions are also introduced in this module.

Unit 4.1 Sets, Relations and Functions

Study Time: 12 Hours Max. Marks: 4

Pre-requisites: Number system.

4.1.1 Sets, Relations and Functions

- Sets and their representation
- Finite and infinite sets
- Empty set, equivalent and equal sets, sub-sets, universal set
- Difference between two sets
- Complement of a set
- Venn diagrams
- Operations on sets (union, intersection and difference)
- Cartesian product of two sets
- Concept of a relation and a function
- Domain, co-domain and range of a function
- Graphs of functions
- Odd and even functions
- Some important functions of the type: modulus function, greatest integer function, polynomial function, logarithmic and exponential functions
- Composition of two or more functions
- Inverse of a function

Supportive video Programme

- Relations and Functions

Unit 4.2 Trigonometric Functions

Study Time: 20 Hours Max. Marks: 8

Pre-requisites: Trigonometric ratios of an acute angle.

4.2.1 Trigonometric Functions-I

- Radian measure of an angle
- Trigonometric functions of the type: $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = a \sin x$, $y = a \cos x$, where x, y are real numbers.
- Graphs of trigonometric functions of the type: $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = a \sin x$, $y = a \cos x$

4.2.2 Trigonometric Functions-II

- Trigonometric functions of $-x$, $\frac{x}{2}$, $x \pm y$, $\frac{\pi}{2} \pm x$, $\pi \pm x$, where x, y are real numbers
- Addition and subtraction formulae for trigonometric functions
- Solution of simple trigonometric equations of the type: $\sin x = 0, \cos x = 0, \tan x = 0, \sin x = \sin\alpha$, $\cos x = \cos\alpha, \tan x = \tan\alpha$

4.2.3 Inverse Trigonometric Functions

- Concept of inverse trigonometric functions
- Properties of inverse trigonometric functions
- Reducing expression in simplest form

4.2.4 Relations between Sides and Angles of a Triangle

- Sine formula, Cosine formula and Projection formula.
- Identities based on these relations

MODULE -5 : CALCULUS

Approach : The word "Calculus " is the Latin name for a stone which was used by the Romans for calculation. Now-a-days it stands for the abbreviation for the words **"Infinitesimal* Calculus"** and is used for calculations with numbers which are very small. With the help of calculus we will be able to solve two types of problems. The first of these involves finding the instantaneous rate at which a variable quantity is changing with respect to another variable. The second type of problem involves finding a function when its rate of change is given. The branches of calculus which respectively deal with these two types of problems are called **"Differential Calculus"** and **"Integral Calculus".**

Applications of Differential Calculus discussed in the lessons on Tangents and Normals and Maxima & Minima. Definite Integrals are used to study area of a bounded region.

**The quantities which can not be readily measured are said to be very small or infinitesimal.*

Unit 5.1 Differential Calculus

Study Time: 38 Hours Max. Marks: 14

Pre-requisites: Trigonometric functions and Exponential and Logarithmic functions.

5.1.1 Limit and Continuity

- Limit of a function(left hand and right hand limits)
- Some standard limits
- Continuity of a function at a point
- Continuity of a function in an interval

Supportive video Programme

- Limit and Continuity

5.1.2 Differentiation

- Derivative of a function at a point
- Geometrical meaning of derivative
- Derivative from first principle
- Derivative of sum, difference, product and quotient of functions and chain rule
- Derivatives of algebraic functions
- Second order derivatives

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- Differentiation and its Applications

5.1.3 Differentiation of Trigonometric Functions

- Differentiation of trigonometric and inverse trigonometric functions from first principle
- Differentiation of trigonometric and inverse trigonometric functions
- Second order derivatives

5.1.4 Differentiation of Exponential and Logarithmic Functions

- Differentiation of Exponential and Logarithmic Functions
- Differentiation by substitution
- Second order derivatives

5.1.5 Tangents and Normals

- Finding equation of tangents and normals to different functions at given point
- Rolle's theorem and Mean Value Theorem (without proof), their validity and applications.

5.1.6 Maxima and Minima

- Monotonic(increasing & decreasing) Functions
- Monotonicity and sign of the derivative
- Maxima and Minima

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- Increasing and Decreasing Functions

Unit 5.2 Integral Calculus

Study Time: 30 Hours Max. Marks: 12

Pre-requisite: Differential Calculus

5.2.1 Integration

- Integration as inverse of differentiation
- Properties of integrals
- Integration by substitution
- Integration by parts
- Some standard integrals of the type:

$$\int\frac{dx}{x^2\pm a^2},\int\frac{dx}{\sqrt{x^2\pm a^2}},\int\frac{dx}{a^2-x^2},\int\frac{dx}{\sqrt{a^2-x^2}},$$

$$\int\frac{dx}{ax^2+bx+c},$$

$$\int\frac{dx}{\sqrt{ax^2+bx+c}},\int\frac{(px+q)dx}{ax^2+bx+c},\int\frac{(px+q)dx}{\sqrt{ax^2+bx+c}},$$

$$\int\sqrt{x^2\pm a^2}\,dx,\int\sqrt{a^2-x^2}\,dx,$$

$$\int e^{ax}\sin bx\,dx,\int e^{ax}\cos bx\,dx,$$

$$\int(px+q)\sqrt{ax^2+bx+c}\;dx,\int\sin^{-1}x\,dx,$$

$$\int\cos^{-1}x\,dx,$$

$$\int\sin^n x\cos^m x\,dx,\int\frac{dx}{a+b\sin x},\int\frac{dx}{a+b\cos x}$$

- Integration using partial fractions

5.2.2 Definite Integrals

- Definite integral as limit of a sum
- Geometrical interpretation of definite integrals
- Fundamental theorem of Integral Calculus (statement only)
- Properties of definite integrals

(i) $$\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$$

(ii) $$\int_a^c f(x)\,dx=\int_a^b f(x)\,dx+\int_b^c f(x)\,dx$$

(iii) $$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

(iv) $$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

(v) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if

$$f(2a-x) = f(x)$$

$$= 0\text{, if } f(2a-x) = -f(x)$$

(vi) $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if f is an even function of x

$= 0$, if f is an odd function of x

- Evaluation of definite integrals using properties
- Application of definite integrals in finding area under a curve

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- Definite Integrals
- Application of Definite Integrals

5.2.3 Differential Equations

- Differential equation, its order and degree
- Formation of a differential equation
- Solutions of first order, first degree differential equations of the type:

(i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x).g(y)$

(iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)y = Q(x)$

- Solution of the differential equation of the form: $\frac{d^2y}{dx^2} = f(x)$
- Particular solution of a differential equation for given conditions

Supportive video Programme

- Differential Equations

MODULE -6 : STATISTICS

Approach: Statistics is the science which deals with collection, representation, interpretation and analysis of the data collected for complete population. The word statistics have been derived from the latin word **"Status "** or the Italian word **"Statista"** or the German word **"Statistik"** , each of which means political state. In ancient days the heads of the tribes or the kings used to collect information with regard to the population of their tribe or kingdom in order to collect taxes to be utilized for the welfare of their tribes or kingdom. The theoretical development of the modern statistics was originated with the introduction of " Theory of Probability" and "Theory of Games of Chances". This module has been designed to develop these concepts in the units on *"Measures of Dispersion"* and *"Random Experiments and Probability"*. These concepts are used in solving daily life problems.

Unit 6.1 Measures of Dispersion

Study Time: 6 Hours Max. Marks: 3

Pre-requisite: Basic concepts of probability, Mean of grouped data, median of ungrouped data

6.1.1 Measures of Dispersion

- Range
- Mean deviation
- Variance and Standard deviation

Unit 6.2 Random Experiments and Probability

Study Time: 12 Hours Max. Marks: 4

Pre-requisite: Basic concepts of probability, Mean, mode, median of grouped and ungrouped data

6.2.1 Random Experiments and Events

- Random experiments
- Sample space of an event
- Types of events, viz. mutually exclusive events and equally likely events

6.2.2 Probability

- Concept of probability
- Use of Permutation and combination in probability
- Addition and multiplication theorems on probability
- Conditional Probability and independent events

Supportive video Programme

- Random Experiments and Probability

OPTIONAL MODULES

(You have to choose any one of the following two modules)

MODULE-7: VECTORS AND THREE DIMENSIONAL GEOMETRY

Approach: If a station B is at a distance of x km from another station A and if a third station C is at a distance of y km from B , then C is not necessarily at a distance of (x + y)km from A. It will occur in very special circumstances only. It depends upon the direction of the line segments from A to B and then from B to C.The quantities with magnitude and direction, known as vectors. Algebra of vectors, dot and cross products of two vectors will be discussed in the unit on ***"Vectors".***

A triangle, a quadrilateral ,a circle, an ellipse are examples of geometric figures in a plane(two dimension) We have also acquainted with objects like a ball ,an ice cream, a cylinder, a cone. None of these is an object in a plane. In order to study the Geometry of these objects, we have to extend our study to three dimensions (space).The study includes the general equation of a plane, the equations of line in symmetrical form and the general equation of a sphere. These concepts are introduced in the unit on "Three Dimensional Geometry".

Unit 7.1 Vectors

Study Time: 6 Hours Max. Marks: 5

Pre-requisites: Plane Geometry, Coordinate Geometry and Trigonometry

7.1.1 Introduction to Vectors

- Need of mentioning direction
- Scalars and vectors
- Vectors as directed line segments
- Magnitude and direction of a vector
- Null vector and Unit vector
- Equality of vectors
- Position vector of a point
- Addition and subtraction of vectors and their properties
- Multiplication of a vector by a scalar and its properties
- Resolution of a vector in two dimensions.
- Resolution of a vector in three dimensions
- Section formula
- Scalar(dot) product of two vectors
- Vector(cross) product of two vectors

Unit 7.2 Three Dimensional Geometry

Study Time: 20 Hours Max. Marks: 10

Pre-requisites: Coordinate Geometry and Trigonometry

7.2.1 Introduction to Three Dimensional Geometry

- Coordinates of a point in space.
- Distance between two points.
- Coordinates of a point which divides a line segment joining two points internally and externally in a given ratio.
- Direction cosines, direction ratios and projection.
- Conditions of parallelism and perpendicularity of two lines.

7.2.2 The Plane

- General equation of a plane.
- Equation of a plane passing through three points.
- Equation of a plane in the normal and intercept form.
- Angle between two planes.
- Plane bisecting angles between two planes.
- Homogeneous Equation of second degree representing two planes.
- Area of a triangle

7.2.3 The Straight Line

- Equation of a straight line in symmetrical form.
- Deduction of the general equations into symmetrical form.
- Perpendicular distance of a point from a straight line.
- Angle between a line and a plane.
- Condition of coplanarity of two lines.

7.2.4 The Sphere

- Equation of a sphere: general form.
- Equation of a sphere through four non coplanar points.
- Diameter form of the equation of a sphere
- Intersection of a sphere and a plane
- Sphere through a given circle

MODULE-8: MATHEMATICS FOR COMMERCES, ECONOMICS AND BUSINESS

Approach: Sharing and helping each other is one of the important aspects of human life. It happens in the case of sharing knowledge, sharing happiness etc. This also happens in the case of starting an Industry or Business on a large scale. It requires huge amount of capital, professional skills and other resources. Thus, a group of people get together and set up a company registered under Companies Act. In this effort, risk is also involved and therefore arises the need for insurance. The Government has a social responsibility towards the nation and has also to look after the security and development of the country. This can not be done without resources which have to be mobilized primarily through the process of taxation-direct and indirect. Application of Mathematics in Industry, Trade, Commerce, Economics, Insurance and Taxation are discussed in this module.

Unit 8.1 Mathematics For Business

Study Time: 20 Hours Max. Marks: 10

Pre-requisites: Commercial Mathematics of Secondary level

8.1.1 Shares and Debentures

- Shares and Dividends
- Debentures
- National Index, BSE Index & Price Index

8.1.2 Index Numbers

- Index numbers : definition and uses
- Construction of index numbers
- Aggregate method
- Simple average of relatives method

8.1.3 Insurance

- Policies and premiums
- General Insurance/Life Insurance- Social Security etc.
- Medical Insurance
- Insurance of movable and immovable properties, factories, goods in transit, vehicles, fire, natural calamities.

8.1.4 Indirect Taxes

- Tax at sources(excise Duty)
- Production Linked Tax
- Value Added Tax(VAT)
- Customs Duty, Import, Export

Unit 8.2 Application of Calculus in Commerce and Economics

Study Time: 6 Hours Max. Marks: 5

Pre-requisites: Calculus

8.2.1 Application of Calculus in Commerce and Economics

- Total cost, Average cost and Marginal cost
- Total revenue, Average revenue and Marginal revenue
- Calculation of Total cost/Revenue for given Marginal cost/Revenue
- Maximization of total revenue and total profits
- Minimization of total cost

SCHEME OF EVALUATION

The learners will be evaluated through Public Examination and through continuous and comprehensive evaluation in the form of Tutor Marked Assignments (**TMA's**).

Mode of Evaluation	**Duration**	**Marks**	**Paper**
Public/Final Examination	3 Hours	100	1
TMA-I **OR** TMA-II	Self -paced	25	
TMA -III (**compulsory**)	Self -paced	25	

The **awards/grades** of TMA will be reflected in the marksheet separately. This will not be considered for inclusion in overall grading in the Public Examination.

Mathematics

गणित

(311)

Assignment - I

मूल्यांकन पत्र - I

(Lessons 1-11)

(पाठ 1 से 11 तक)

Max. Marks: 25

कुल अंक : 25

Note: (i) All questions are compulsory. Each question carries equal marks.

टिप्पणी: सभी प्रश्नों के उत्तर देने अनिवार्य हैं। प्रत्येक प्रश्न के अंक समान हैं।

(ii) Write your name, enrolment number, AI name and subject etc. on the top of the first page of the answer sheet.

उत्तर पुस्तिका के प्रथम पृष्ठ पर ऊपर की ओर अपना नाम, अनुक्रमांक, अध्ययन केन्द्र का नाम, विषय आदि स्पष्ट शब्दों में लिखिए।

1. Answer any two of the following questions:
निम्नलिखित प्रश्नों में से किन्हीं **दो** के उत्तर दीजिए:

(i) If $\sqrt{3}+i=(a+ib)(c+id)$, then find the value of the following .

$\sqrt{3}+i=(a+ib)(c+id)$, तो निम्न का मान ज्ञात कीजिए।

$$\tan^{-1}\left(\frac{b}{a}\right)+\tan^{-1}\left(\frac{d}{c}\right)$$

(ii) If the quadratic equations $x^2-ax+b=0$ and $x^2+bx-a=0$ have a common root, then show that a-b=1

यदि द्विघाती समीकरण $x^2-ax+b=0$ तथा $x^2+bx-a=0$ का एक उभयनिष्ठ मूल है, तो दर्शाइए कि a-b=1

(iii) A point moves in a plane such a way that the sum of its distances from the point (ae,o) and (-ae,o) is 2a, show that the locus of this point $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

एक बिंदु एक तल में ऐसे चलता है कि इसके बिंदु (ae,o) तथा (-ae,o) से दूरियों का योग 2a है, दर्शाइए कि इस बिंदु का बिंदुपथ $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$ है।

2. Answer any two of the following questions:
निम्नलिखित प्रश्नों में से किन्हीं **दो** के उत्तर दीजिए:

(i) Find the equation of the line through the intersection of the lines 2x+3y-4=0 and x-5y+7=0 and has x-intercept equal to -4.

उस रेखा का समीकरण ज्ञात कीजिए जो रेखाओं $2x+3y-4=0$ तथा $x-5y+7=0$ के प्रतिच्छेद बिन्दु से होकर जाती है तथा जिसका x अन्तः खंड -4 के बराबर है।

(ii) m parallel lines in a plane are intersected by a family of n parallel lines. Find the total number of parallelograms so formed.

किसी समतल में m समान्तर रेखाओं को एक अन्य n समान्तर रेखाओं का कुल प्रतिच्छेद करता है। इस प्रकार बनने वाले समान्तर चर्तुभुजों की संख्या ज्ञात कीजिए।

(iii) If P and Q are respectively the sum of the odd and even terms in the expansion of $(x+1)^n$, then prove that $P^2-Q^2=(x^2-1)^n$

यदि $(x+1)^n$ के प्रसार में विषम पदों तथा सम पदों का योग क्रमशः P तथा Q है, तो सिद्ध कीजिए कि $P^2-Q^2=(x^2-1)^n$

3. Solve graphically the system of inequations:

निम्न असमीकरण निकाय का हल आलेखीय विधि द्वारा कीजिए:

$$5x + 3y < 0 \; ; \; 3x - 4y + 7 > 0$$

Or (अथवा)

Find the matrix A satisfying the following matrix equation:

निम्न आव्यूह समीकरण को सन्तुष्ट करने वाला आव्यूह A ज्ञात कीजिए:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Using properties of determinants, prove the following

सारणिकों के गुण धर्मों का प्रयोग कर निम्न को सिद्ध कीजिए।

$$\begin{bmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{bmatrix} = 1+a^2+b^2+c^2$$

Or

If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, Find A^{-1}, Using A^{-1}, solve the system of equations

$2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$

यदि $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ है तो A^{-1} ज्ञात कीजिए.

A^{-1} का प्रयोग कर निम्न समीकरण निकाय को हल कीजिए:

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

5. **Project work (परियोजना कार्य)**

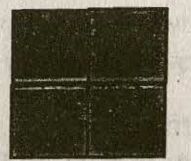
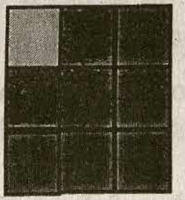
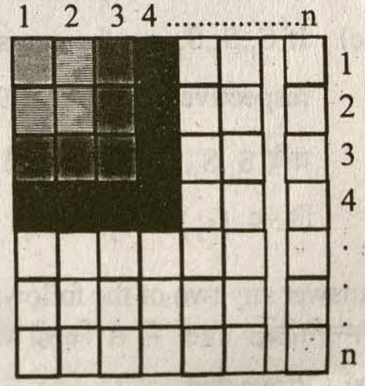
To experimentally show the sum of first n odd natural numbers, i.e.

प्रयोग द्वारा प्रथम n विषम संख्याओं का योग ज्ञात करना अर्थात,

1+3+5+7 .... = ...

Let us form the following table

आइए हम निम्न तालिका बनाँए,

Odd natural Numbers **विषम संख्याए**	**Sum** **योग**	**Number of terms** **पदों की संख्या**	**Graphical representation** **आलेखीय निरूपण**
1	$1=1^2$	1	$: 1 = 1^2$
1,3	$4=2^2$	2	 $: 4 = 2^2$
1,3,5	$9=3^2$	3	 $: 9 = 3^2$
1,3,5,7	$16=4^2$	4	$: 16 = 4^2$
1,3,5,7 terms	1+3+5+7--- n terms $= n^2$	n	 1 2 3 4 ....n 1 2 3 4 . . . n n^2

Mathematics

गणित

(311)

Assignment - II

मूल्यांकन पत्र - II

(Lessons 12-21)

(पाठ 12 से 21 तक)

Max. Marks: 25
कुल अंक : 25

Note: (i) All questions are compulsory. Each question carries equal marks.
टिप्पणी: सभी प्रश्नों के उत्तर देने अनिवार्य हैं। प्रत्येक प्रश्न के अंक समान हैं।

(ii) Write your name, enrolment number, AI name and subject etc. on the top of the first page of the answer sheet.
उत्तर पुस्तिका के प्रथम पृष्ठ पर ऊपर की ओर अपना नाम, अनुक्रमांक, अध्ययन केन्द्र का नाम, विषय आदि स्पष्ट शब्दों में लिखिए।

1. Answer any two of the following questions:
निम्नलिखित प्रश्नों में से किन्हीं **दो** के उत्तर दीजिए:

(a) The N arithmetic means, A_1, A_2, A_3, ------, A_n between 30 and 90 are such that the ratio of A_1 to A_n is 3:7 find the value of n.
30 और 90 के मध्य n समांतर माध्य A_1, A_2, A_3, ------, A_n इस प्रकार हैं कि A, तथा A_n का अनुपात 3:7 है। n का मान ज्ञात कीजिए।

(b) The product of first three terms of a G.P. is 5832, If 14 is added to the second term and 4 is added to its third term, the three terms form an A.P. Find the terms of the G.P.
एक गुणोत्तर श्रेणी (G..P) के प्रथम तीन पदों का गुणनफल 5832 है। यदि इसके दूसरे पद में 14 तथा तीसरे पद में 4 जोड़ दिया जाए तब यह तीनों पद समांत्तर श्रेणी (A.P) बनाते हैं। G.P. के पद ज्ञात कीजिए।

(c) If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that $9\ (s_2)^2 = s_3(1+8s_1)$
यदि S_1, S_2, S_3 क्रमशः पहली प्राकृत संख्याओं, उनके वर्गों, उनके धनों के योग हैं ,तो दर्शाइए कि $9\ (s_2)^2 = s_3(1+8s_1)$

2. Answer any two of the following questions:
निम्ननिखित प्रश्नों में से किन्हीं **दो** के उत्तर दीजिए:

(a) Prove that
$\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A \cos B = \sin^2 A$.
सिद्ध कीजिए कि:
$\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A \cos B = \sin^2 A$.

(b) If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, show that $fof(x) = x$ for all $x \neq \frac{2}{3}$. What is the inverse of f.?

यदि $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$, है तो दर्शाइए कि सभी $x \neq \frac{2}{3}$, के लिए $fof(x) = x$. फलन f का प्रतिलोम फलन क्या है?

(c) Find the equation of the ellipse whose focus is (1, -2), the directrix 3x-2y+5=0 and eccentricity equal to $e = \frac{1}{2}$.

उसी दीर्घवृत का समीकरण ज्ञात कीजिए जिसका केन्द्र (1, -2) तथा नाभि उत्केन्द्रता $e = \frac{1}{2}$ है तथा नियता $3x - 2y + 5 = 0$ है।

3. Determine the values of a,b,c for which the function.

$$f(x) = \begin{cases} \frac{\text{Sin}(a+1)x + \sin x}{x} & \text{, for } x < 0 \\ c & \text{, for } x = 0 \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & \text{, for } x > 0 \end{cases} \text{ is continuous at } x = 0$$

a,b,c के वह मान ज्ञात कीजिए जिनके लिए

$$f(x) = \begin{cases} \frac{\text{Sin}(a+1)x + \sin x}{x} & , x < 0 \text{ के लिए} \\ c & , x = 0 \text{ के लिए} \\ \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \text{ के लिए} \end{cases}$$

$x = 0$ पर सतत् है।

or

In a Δ ABC, Prove that

$a^3. \cos(B-C) + b^3. \cos(C-A) + c^3. \cos(A-B) = 3\,abc.$

Δ ABC में, सिद्ध कीजिए कि–

$a^3. \cos(B-C) + b^3. \cos(C-A) + c^3. \cos(A-B) = 3\,abc.$

4. Answer any two of the following questions.
निम्नलिखित प्रश्नों में से किन्हीं दो के उत्तर दीजिए:

(a) Find the derivative of $(2x+3)^3$ using first principles.
प्रथम सिद्धान्त का प्रयोग करते हुए $(2x+3)^3$ का अवकलज ज्ञात कीजिए।

(b) Solve the equation: $4\sin x \sin 2x \sin 4x = \sin 3x$
निम्न समीकरण को हल कीजिए:
$4\sin x \sin 2x \sin 4x = \sin 3x$

(c) Solve for x
$2 \tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$
x के मान के लिए हल कीजिए।
$2 \tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

5. **Project Work**

Given below is a graphical representation, which depicts reasons of strikes/closures in factories/mills. Read the graph and reply the following questions:

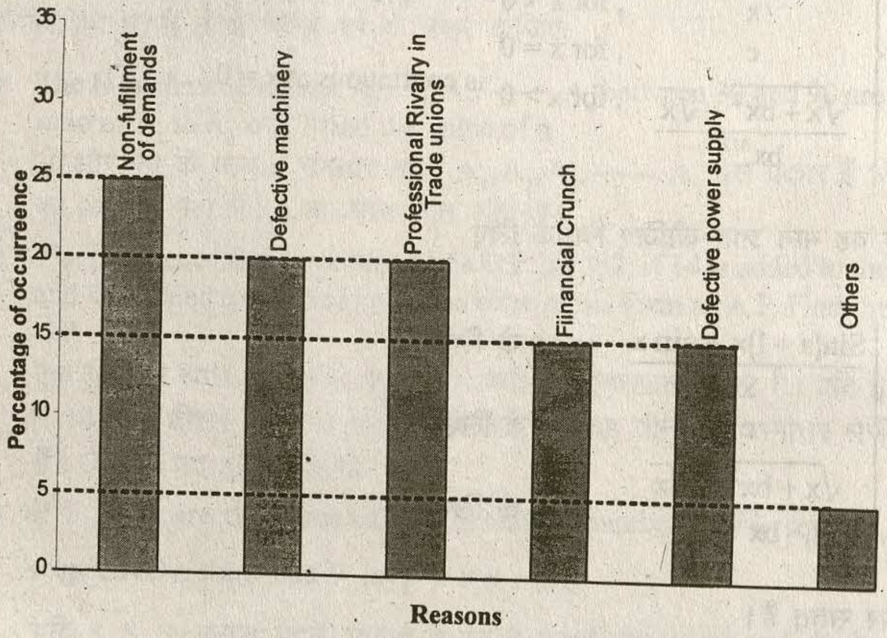


Fig. 1
(आकृति 1)

(i) According to you, which two are the main reasons for closures of factories.

(ii) Do you see any other important reasons which has not been visualised by the experimenter? If yes, state that (those).

(iii) Do you think the extent of occurrence of strikes / closures of mills is correct? If not, according to you for which reason(s), it is not correct.

(iv) Plan a visit, if possible to a group of sick factories, and collect data yourself and then draw the graph.

परियोजना कार्य

ऊपर दी गई आकृति 1 में विभिन्न फैक्ट्री (अथवा मिलों) के बन्द होने (हड़ताल होने) के कारण दर्शाए गये है। आकृति को देखकर निम्न प्रश्नों के उत्तर दीजिए।

1. आपके अनुसार मिलों के बन्द होने के दो मुख्य कारण क्या हैं।
2. क्या आप अन्य कोई ऐसा (ऐसे) कारण सोच सकते हैं जो प्रयोग कर्ता ने छोड़ दिया है? यदि हाँ, तो उसे (उन्हें) लिखिए।
3. क्या आपके अनुसार इन कारणों के घटने की प्रतिशतता सही है।
   यदि नहीं, तो बताइए कि किस (किन) कारणों की प्रतिशतता सही नहीं है।
4. यदि हो सके तो ऐसी न ठीक चलती (ज़ो बन्द होने के कगार पर) मिलों के समूह में जाकर उनके ठीक न चलने के कारण स्वयं एकत्रित कर उनका आलेखीय निरूपण कीजिए।

Mathematics

गणित

(311)

Assignment - III

मूल्यांकन पत्र - III

(Lessons 22-31)

(पाठ 22 से 31 तथा अन्य पाठ-वैकल्पिक मॉड्यूल्स)

Max. Marks: 25
कुल अंक : 25

Note: (i) All questions are compulsory. Each question carries equal marks.
टिप्पणी: सभी प्रश्नों के उत्तर देने अनिवार्य हैं। प्रत्येक प्रश्न के अंक समान हैं।

(ii) Write your name, enrolment number, AI name and subject etc. on the top of the first page of the answer sheet.
उत्तर पुस्तिका के प्रथम पृष्ठ पर ऊपर की ओर अपना नाम, अनुक्रमांक, अध्ययन केन्द्र का नाम, विषय आदि स्पष्ट शब्दों में लिखिए।

1. Answer any two of the following questions:
निम्न में से किन्हीं **दो** प्रश्नों के उत्तर दीजिए:

(a) If $y = x^x$, then prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

यदि $y = x^x$ है तो सिद्ध कीजिए कि $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

(b) It is given that for the function f given by $f(x) = x^3+bx^2+ax$, $x \in [1,3]$ Roll's theorm holds with $c = 2 + \frac{1}{\sqrt{3}}$, find the values of a and b.

यदि $x \in [1,3]$ पर फलन f प्रदत्त है $f(x) = x^3+bx^2+ax$, द्वारा, के लिए रोले का प्रमेय सत्यापित है, और $c = 2 + \frac{1}{\sqrt{3}}$ हो, तो a तथा b के मान ज्ञात कीजिए।

(c) If $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$, show that $\frac{dy}{dx} = \sqrt{a^2 - x^2}$

यदि $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$, है तो दर्शाइए कि $\frac{dy}{dx} = \sqrt{a^2 - x^2}$

2. Answer any two of the following questions:
निम्न में से किन्हीं **दो** प्रश्नों के उत्तर दीजिए:

(a) Show that the function f(x) = tan x - 4x is decreasing function on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

दर्शाइए कि $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ पर फलन $f(x) = \tan x - 4x$ एक ह्रासमान फलन है।

(b) Show that the greatest volume of the cylinder which can be inscribed in a cone of height h and semivertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$

दर्शाइए कि h ऊँचाई तथा α अर्धशीर्ष कोण वाले शंकु के अन्तर्गत बने अधिकतम आयतन वाले बेलन का आयतन $\frac{4}{27}\pi h^3 \tan^2 \alpha$ है।

(c) Find the equation of tangent line to $y=2x^2+7$ which is parallel to the line $4x-y+3=0$

वक्र $y=2x^2+7$ की उस स्पर्श रेखा का समीकरण ज्ञात कीजिए जो कि सरलरेखा $4x-y+3=0$ के समांतर हो।

3. (a) Evaluate

मान ज्ञात कीजिए: $\int \frac{(x-4)e^x}{(x-2)^3} dx$

Or

Evaluate:

मान ज्ञात कीजिए: $\int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$

Or (अथवा)

(b) Find the general solution of the following differential equation.

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$$

निम्न अवकल समीकरण का व्यापक हल ज्ञात कीजिए।

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$$

4. (a) Find the equation of the plane passing through the point (-1,3,2) and perpendicular to each of the plane $x+2y+3z=5$ and $3x+3y+z=0$

बिन्दु (-1,3,2) से होकर जाने वाले उस समतल का समीकरण ज्ञात कीजिए जो समतलों $x+2y+3z=5$ तथा $3x+3y+z=0$ में से प्रत्येक पर लम्ब हो।

Or (अथवा)

Calculate premium for a policy of Rs. 3,00,000 endowment insurance (with profits), if a man aged 29 years wants it for a term of 25 years, assume that the man pays through salary saving scheme and the policy also covers the risk of accident.

3,00,000 रू० की अक्षय निधि (एन्डौमेंट) बीमा पॉलिसी (लाभ सहित) के लिए प्रीमियम की गणना कीजिए, यदि कोई 29 वर्ष आयु का व्यक्ति उसे 25 वर्ष के लिए लेना चाहे। यह मान लीजिए कि वह व्यक्ति वेतन बचत योजना के अंतर्गत भुगतान करेगा तथा पॉलिसी दुर्घटना का लाभ भी समाहित करती है।

(b) The means of two samples of sizes 50 and 100 respectively are 54.1 and 50.3; the standard deviations are 8 and 7. Find the standard deviation of the sample of size 150.

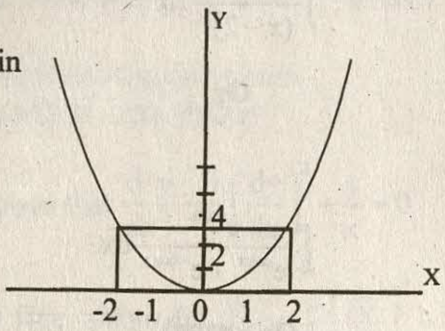
दो प्रतिदर्शों के मापों 50 और 100 के माध्य क्रमशः 54.1 और 50.3 हैं तथा मानक विचलन 8 और 7 हैं। दोनो प्रतिदर्शों को इकट्ठा करने पर 150 माप वाले प्रतिदर्श का मानक विचलन ज्ञात कीजिए।

5. Project work

To find out experimentally if a function is increasing or decreasing or none in an open interval

Consider $f(x) = x^2, x \in R$. Take the origin as the point. The graph of f(x) is as follows.

Take values left to origin as $-2, \frac{-3}{2}, -1, \frac{-1}{2}, 0$



Take values right to origin as $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

Fig. 1

Let us form the table of values of f(x) at these points

Values left to origin

x	f(x)
-2	4
-3/2	9/4
-1	1
$\frac{1}{2}$	$\frac{1}{4}$
0	0

Table-1

values right to origin

x	f(x)
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
3/2	9/4
2	4

Table-2

consider the graph to the right of origin

As x increases, y increases, for $x > 0$, the function is increasing [as. $x_2 > x_1$ .f $(x_2) > f(x_1)$]

To the left of origin

As x decreases, y increases, i.e. $x_2 < x_1$, $f(x_2) > f(x_1)$

$\Rightarrow$ for $x < 0$, the function is decreasing.

5. परियोजना कार्य

प्रयोग द्वारा ज्ञात करना कि एक फलन एक खुले अन्तराल में वर्धमान है, ह्रासमान है अथवा कुछ नहीं

माना $f(x) = x^2, x \in R$. माना मूल बिन्दु लें। f(x) का आलेख नीचे दिया है।

मूल बिन्दु से बायें ओर के कुछ बिन्दु लें: $-2, \frac{-3}{2}, -1, \frac{-1}{2}, 0$

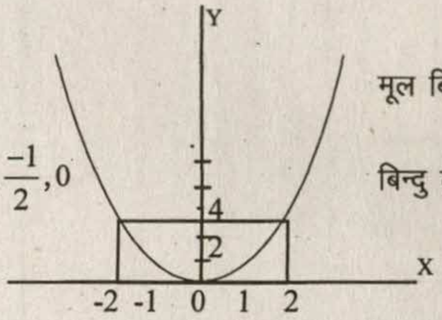


Fig. 1

मूल बिन्दु से दायें ओर के बिन्दु लें: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

आइए f(x) के उपरोक्त मानों की तालिकाए बनाएँ

x	f(x)
-2	4
-3/2	9/4
-1	1
$\frac{1}{2}$	$\frac{1}{4}$
0	0

Table-1

x	f(x)
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
3/2	9/4
2	4

Table-2

माना आलेख मूल बिन्दु से दायीं ओरः जैसे जैसे x-बढ़ता है, y भी बढ़ता है

अतः $x > 0$ के लिए फलन वर्धमान है $[x_2 > x_1, f(x_2) > f(x_1)]$

आलेख मूल बिन्दु से बायें ओर देखें। जैसे जैसे x, बढ़ता है y घटता है

अतः $x < 0$ के लिए फलन ह्रासमान है। $[x_2 < x_1, f(x_2) > f(x)]$

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		Easy	Difficult	Interesting	Confusing	Simple	Complex	Useful	Not useful	Very helpful	Somewhat helpful	Not helpful
1.												
2.												
3.												
4.												
5.												
6.												
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11.												
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Third fold

Feed back on Questions

Lesson no.	Lesson Name	Intext Questions		Terminal Question		
		Useful	Not useful	Easy	Diff.	V. diff.
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2.						
3.						
4.						
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FUNDAMENTAL DUTIES

Part IVA (Article 51 A)

It shall be the duty of every citizen of India-

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forest, lake, rivers and wild life, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that nation constantly rises to higher levels of endeavour and achievement.

A brief Guide to NIOS web site

The success of open learning and distance education very much depends upon the harnessing of the new and latest technology. The emerging Internet and Web technology help in effective dissemination of knowledge breaking all geographical boundaries. The web-site is a dynamic source of latest information and is also electronic information guide. The contents in the NIOS web site are open to all.

The learners can have an access to NIOS web-site at the following address:

http:/www.nos.org & nios.ac.in

Clicking the site address will bring the user to NIOS home page that will further guide them to visit different information pages of NIOS. NIOS is also developing a school network through Internet known as Indian Open Schooling Network (IOSN). The network will provide a common communication platform for learners and educators. NIOS is offering Certificate in Computer Applications (CCA) through selected AVI. This course is also offered through Internet on NIOS Web-Site.

NOT FOR SALE